Chapter 9

Radiative transfer in planetary atmospheres

Warning: This chapter is just a very rough introduction to planetary atmospheres.

Since the discovery of planets around other stars than the sun, the issue of how atmospheres of such "exoplanets" would look when observed remotely has risen to the forefront of astrophysics. From the Earth's atmosphere we can learn of course a lot, but it is not trivial to predict what an atmosphere of arbitrary composition and structure would look like.

Let us first discuss some general considerations and then focus on some of the technical issues involved.

9.1 Hydrostatic equilibrium

Suppose we have a rocky planet of mass M_p and radius R_p, and we study only a thin atmosphere on top. We take z to be the coordinate such that $z = 0$ means the surface. By assuming the atmosphere to be geometrically thin we assume that $z \ll R_p$. The hydrostatic equilibrium equation is then

$$\frac{dp(z)}{dz} = -g\rho(z) \tag{9.1}$$

where $p(z)$ is the pressure and $\rho(z)$ the density of the gas. The gravitational surface acceleration g is assumed to be constant by virtue of $z \ll R_p$. We can integrate this equation from some value z to "∞":

$$\int_z^\infty \frac{dp(z')}{dz'}dz' = -g\int_z^\infty \rho(z')dz' \tag{9.2}$$

Which leads to

$$p(z) = g\sigma(z) \tag{9.3}$$

with $\sigma(z)$ the column of gas above z:

$$\sigma(z) = \int_z^\infty \rho(z')dz' \tag{9.4}$$

This means that the pressure experienced by some parcel of gas does not change even if we heat up the atmosphere: only its *location* z will change. Because of this phenomenon it may be more pragmatic to use σ or equivalently p as a vertical coordinate than z. Indeed, in atmospheric physics you will see usually the pressure p as a vertical coordinate rather than z. Note that p increases when z decreases and vice versa. So with this new philosophy (having p as a coordinate), now z is a function of p. The hydrostatic equilibrium equation (Eq. 9.1) now becomes

$$\frac{dz(p)}{dp} = -\frac{1}{g\rho(p)} \tag{9.5}$$

which integrates to

$$z(p) = -\int_{p_{\mathrm{surf}}}^{p} \frac{dp'}{g\rho(p')} \tag{9.6}$$

where p_{surf} is the surface pressure (pressure at $z = 0$).

For most atmospheres we can take the ideal gas law as a good approximation of the equation of state:

$$p = \frac{\rho k_B T}{\mu} = \rho c_s^2 \tag{9.7}$$

where μ is the mean molecular weight and c_s is the *isothermal sound speed*. For the Earth's atmosphere this is $\mu \simeq 29\, m_p$ with m_p the proton mass.

If we have, as the simplest example, an isothermal atmosphere (T =constant), then the hydrostatic equilibrium equation in the form Eq. (9.1) integrates to

$$p(z) = p_{\mathrm{surf}} \exp\left(-\frac{z}{H_p}\right) \tag{9.8}$$

with the pressure scale height H_p given by

$$H_p = \frac{c_s^2}{g} = \frac{k_B T}{g\mu} \tag{9.9}$$

Equivalently, the hydrostatic equilibrium equation in the form Eq. (9.6) integrates to

$$z(p) = -H_p\,(\ln p - \ln p_{\mathrm{surf}}) \tag{9.10}$$

An isothermal atmosphere is therefore an exponential atmosphere.

In general planetary atmospheres are far from isothermal. It is in fact one of the challenges we will discuss in this chapter how to compute $T(z)$ or equivalently $T(p)$.

9.2 Radiative energy transport in an atmosphere

9.2.1 Basic two-stream transport equations

Let us focus for now on the radiative transfer equations for the thermal balance of an atmosphere. We have already seen in Section 3.7 how the basics of radiative transfer in plane parallel atmospheres work. We have also seen in Section 4.4.2 that a reasonably good approximation is the two-stream approximation. Let us write out these equations under the simple assumption that scattering is not important:

$$\frac{1}{\sqrt{3}}\frac{dI_{+,\nu}}{dz} = \rho\kappa_\nu[B_\nu(T(z)) - I_{+,\nu}] \tag{9.11}$$

$$-\frac{1}{\sqrt{3}}\frac{dI_{-,\nu}}{dz} = \rho\kappa_\nu[B_\nu(T(z)) - I_{-,\nu}] \tag{9.12}$$

The mean intensity is then

$$J_\nu(z) = \frac{I_{+,\nu}(z) + I_{-,\nu}(z)}{2} \tag{9.13}$$

while the netto flux is

$$F_\nu(z) = \frac{2\pi}{\sqrt{3}}[I_{+,\nu}(z) - I_{-,\nu}(z)] \tag{9.14}$$

With this definition of the flux we can derive the following differential equation for $F_\nu(z)$:

$$\frac{dF_\nu(z)}{dz} = 4\pi\rho\kappa_\nu[B_\nu(T(z)) - J_\nu(z)] \tag{9.15}$$

which is simply a way of writing flux conservation.

If we have radiative equilibrium at every z, then we have

$$\int_0^\infty \kappa_\nu J_\nu d\nu = \int_0^\infty \kappa_\nu B_\nu(T)d\nu \equiv \kappa_P(T)\frac{\sigma_{\rm SB}}{\pi}T^4 \tag{9.16}$$

For the frequency-integrated flux $F(z) = \int_0^\infty F_\nu(z)d\nu$ this means

$$\frac{dF(z)}{dz} = 0 \tag{9.17}$$

i.e. we would have a constant flux. If the planet has a netto heat source from below, then this sets the flux at the bottom of the atmosphere, and the rest follows by $F(z) = constant$. A very young planet which is still cooling down would be such a case.

However, for older planets such as the Earth the flux from below is negligibly small. The above equations would thus predict $F(z) = 0$, and hence $T(z) = 0$. Clearly this is not the case. The reason: Irradiation by the Sun.

9.2.2 Irradiation by the star

Since the stellar (solar) radiation field is highly an-isotropic, it might not be a good idea to treat it with the two-stream approximation. The easiest way to include irradiation by the Sun is to treat the Sun's light as a separate component of the radiation field. Since radiative transfer is a linear theory, you can always split the intensity I_ν into

$$I_\nu = I_\nu^* + I_\nu^{at} \tag{9.18}$$

where I_ν^* is taken to be the stellar radiation and I_ν^{at} the radiation field produced by the atmosphere itself through thermal emission.

Let the stellar flux as seen at the planet's location be

$$F_\nu^* = \frac{L_\nu^*}{4\pi d^2} \tag{9.19}$$

where d is the distance of the planet (e.g. Earth) to the star (e.g. Sun). This flux enters the atmosphere at some angle φ with the surface, with $\varphi = \pi/2$ meaning that the sun is at zenith and $\varphi = 0$ meaning sunset or dawn. The netto irradiative flux is then

$$F_\nu^{*,\rm irr} = \sin(\varphi)F_\nu^* = \sin(\varphi)\frac{L_\nu^*}{4\pi d^2} \tag{9.20}$$

Since the flux experiences extinction due to absorption by the atmosphere, we can write

$$F_\nu^{*,\rm irr}(z) = F_\nu^{*,\rm irr}\exp\left(-\frac{\tau_\nu(z)}{\sin\varphi}\right) \tag{9.21}$$

where $\tau_\nu(z)$ is the vertical optical depth from z to infinity (more precisely: to $z = z_{\rm max}$):

$$\tau_\nu(z) = \int_z^\infty \rho\kappa_\nu dz' \tag{9.22}$$

The radiative equilibrium equation (Eq. 9.16) now becomes

$$\int_0^\infty \kappa_\nu J_\nu d\nu + \frac{1}{4\pi}\int_0^\infty \kappa_\nu F_\nu^* d\nu = \kappa_P(T)\frac{\sigma_{\rm SB}}{\pi}T^4 \tag{9.23}$$

For the frequency-integrated flux this means

$$F(z) + F^{*,\rm irr}(z) = \text{constant} \tag{9.24}$$

For an old planet, without internal heat, this would mean that $F(z) = -F^{*,\rm irr}(z)$.

9.2.3 Diurnal variations and the deviation from radiative equilibrium

Even in an old planet's atmosphere there is not necessarily perfect radiative equilibrium. The Earth rotates and its gets day and night. During the day φ changes. If at every moment in time there were to be radiative equilibrium we would freeze to death during the night and become overheated during the day. The air in the atmosphere is a certain heat capacity, meaning that it takes a while before enough radiative energy is absorbed to adapt the temperature to the radiative equilibrium value. Often this time scale is longer than the diurnal variation of the irradiation. Then there are also the hydrodynamic circulations of air over the planet's surface, allowing heat to be transported not only via radiation but also via gas flows. We will here, however, not discuss this further.

A reasonable estimate of the average temperature structure of the atmosphere can be obtained by inserting an average irradiation flux at the top.

9.3 A simple 1-layer model with 2 grey opacities

Suppose we have an atmosphere that is completely transparent at all wavelengths, except in one well-defined layer located at some pressure p with with width $\Delta p > 0$. This layer corresponds to a $\Delta\sigma$ in the following way:

$$\Delta\sigma = \frac{\Delta p}{g} \tag{9.25}$$

where both $\Delta\sigma$ and Δp are defined to be > 0. Now let us assume that at stellar wavelengths the absorption opacity is κ_* and at the infrared wavelength emitted by the atmosphere the absorption opacity is κ_{at}. For both cases we treat the opacity as grey, i.e. we assume that the opacity effectively is a step function, being $\kappa_\nu = \kappa_*$ for $\lambda \leq 2\mu$m and $\kappa_\nu = \kappa_{\mathrm{at}}$ for $\lambda > 2\mu$m, where 2μm is our estimate of the boundary between "stellar" and "atmospheric" photons. For convenience we assume that the scattering opacity is always zero. *We also write all fluxes as positive, whether they point upward or not; the context should make clear in which direction they point*. The star irradiates the atmosphere under an angle φ with projected frequency-integrated irradiation flux $F_{*,\mathrm{irr}} = \sin(\varphi)F_*$. Below the layer, only

$$F^{\mathrm{below}}_{*,\mathrm{irr}} = F_{*,\mathrm{irr}}e^{-\tau_*} \tag{9.26}$$

of that radiation is left to irradiated the surface of the planet, with

$$\tau_* = \frac{\Delta\sigma\kappa_*}{\sin(\varphi)} \tag{9.27}$$

We assume that the planet's surface absorbs this radiation fully, and re-radiates it as a Planck function at the temperature of the surface T_s, so that the surface flux upward is:

$$F_s = \sigma_{\mathrm{SB}}T_s^4 \tag{9.28}$$

The layer apparently absorbs

$$Q^*_+ = F_{*,\mathrm{irr}} - F^{\mathrm{below}}_{*,\mathrm{irr}} = F_{*,\mathrm{irr}}\,(1 - e^{-\tau_*}) \tag{9.29}$$

amount of energy per second (unit: erg/s/cm^2) from the stellar radiation. It also absorbs radiation from the surface:

$$Q^s_+ = F_s\,(1 - e^{-\tau_s}) \tag{9.30}$$

where

$$\tau_s = \tau_{\mathrm{at}} = \sqrt{3}\Delta\sigma\kappa_{\mathrm{at}} \tag{9.31}$$

where the $\sqrt{3}$ comes in through the two-stream approximation.

If we assume radiative equilibrium the temperature of the layer must be such that it emits as much as it absorbs. In the two-stream approximation the emitted flux from the layer in both directions is

$$F_{\rm at} = \frac{2}{\sqrt{3}}\sigma_{\rm SB} T_{\rm at}^4 (1 - e^{-\tau_{\rm at}}) \tag{9.32}$$

A more accurate flux would be:

$$F_{\rm at} = \sigma_{\rm SB} T_{\rm at}^4 (1 - e^{-\tau_{\rm at}}) \tag{9.33}$$

but this is only 15% different from the two-layer one. Let us take the more accurate one (Eq. 9.33). Radiative equilibrium in the layer means:

$$2F_{\rm at} = Q_+^* + Q_+^s = F_{*,\rm irr}(1 - e^{-\tau_*}) + F_s(1 - e^{-\tau_s}) \tag{9.34}$$

Radiative equilibrium at the surface means:

$$F_s = F_{\rm at} + F_{*,\rm irr}^{\rm below} = F_{\rm at} + F_{*,\rm irr} e^{-\tau_*} \tag{9.35}$$

Eqs. (9.34,9.35) forms a set of coupled equations. We can eliminate F_s and we solve for $F_{\rm at}$ to obtain

$$F_{\rm at} = F_{*,\rm irr}\left(\frac{1 - e^{-(\tau_* + \tau_{\rm at})}}{1 + e^{-\tau_{\rm at}}}\right) \tag{9.36}$$

Inserting this into Eq. (9.35) we obtain

$$F_s = \left(\frac{1 + e^{-\tau_*}}{1 + e^{-\tau_{\rm at}}}\right) F_{*,\rm irr} \tag{9.37}$$

Now that we know what the flux balance is, we can calculate the corresponding temperatures from Eqs. (9.33, 9.28):

$$T_{\rm at} = \left[\frac{F_{\rm at}}{\sigma_{\rm SB}} \frac{1}{1 - e^{-\tau_{\rm at}}}\right]^{1/4} \tag{9.38}$$

and

$$T_s = \left[\frac{F_s}{\sigma_{\rm SB}}\right]^{1/4} \tag{9.39}$$

This simplest possible model of an atmosphere is really "to be taken with a grain of salt", because it ignores a lot of effects. For instance, it completely ignores scattering, and more importantly, it ignores convection. The latter tends to produce a *troposphere*, where the temperature structure is very close to adiabatic. This example model is therefore really nothing more than a simple toy model to demonstrate the basics of radiative equilibrium in an atmosphere.

9.4 Opacities in planetary atmospheres - a very brief overview

Atmospheric opacities can be divided into continuum opacities due to aerosols, water (or other liquid) droplets and Rayleigh scattering off molecules, and line opacities due to the most common molecules.

As we learned from the chapter on line transfer: lines are generally very thin, covering just a tiny part of the spectrum. Therefore, in the interstellar medium the energy budget is often dominated by continuum opacities due to dust. However, in planetary atmospheres dust is often settled to the surface, and clouds of condensables might at least partially disappear, leaving a clear sky. In that case the lines from molecules take over the energy balance. Because of the high pressures in planetary atmospheres (at least, compared to interstellar medium pressures) the line wings of the Lorentz profiles of the lines are rather broad. Additionally in the infrared there can be literally millions of lines from H_2O, CO_2 and other molecules. All in all the lines therefore cover a substantial part of the spectrum, and thus act in a way similar to what dust continuum opacities would do in lower-density environments.

In the chapter on line transfer we dealt with the case where T_{gas} was known, but the level populations could be non-LTE. In planetary atmospheres we are mostly lucky that the populations are usually in LTE, due to the high density. But the temperature T_{gas} is typically *not* known in advance. The complication of non-LTE is now replaced by the complication of non-radiative-equilibrium. For continuum opacities we know how to solve for the temperature (see Chapter 5). For line transfer we have not yet treated this problem.

In principle we could use similar techniques as we did for the dust continuum. We could, in a way, treat the gas opacity κ_ν as some kind of "dust opacity" and use the same methods as discussed in Chapter 5) to solve for the gas temperature. However, this is not so easy.

First of all, the opacities of the gas κ_ν depend on temperature. For the method of Bjorkman & Wood this is bad. With an iteration scheme one can get around this, but it remains tricky. For Lambda-Iteration-type schemes this is not a problem, and it could easily be done.

But the other problem is more problematic: the shape of κ_ν as a function of ν is so wild, i.e. it changes so dramatically with ν, that one would need millions of frequency points ν_i to sample the opacity appropriately. This would simply be computationally too expensive.

There are several ways by which one can get around this. Here I list a few:

- *Simple Rosseland and Planck means:* The simplest is just to compute the Rosseland and Planck mean opacities. This is not very accurate in atmospheres which are optically thin in certain wavelength ranges.
- *Opacity sampling*: Take a random set of frequencies ν_i and treat the spectrum as being fully defined by these frequencies. If the number of frequency points is large enough, then these points represent sufficiently well the range of opacities encountered.
- *Opacity distribution functions*: We define a set of rather broad ranges in frequency. For each range we sample how many times an opacity between $\kappa + \Delta\kappa$ occurs, so that we have a probability distribution function $p(\kappa)$ for each frequency range.

Details for how radiative transfer in planetary atmospheres is done can be found in:

1. Raymond Pierrehumbert: "Principles of Planetary Climate"
2. Sara Seager: "Exoplanet Atmospheres"