6 Turbulence

- What do we mean when we talk about turbulence? There are two fundamentally different regimes of fluid flow: smooth "laminar" flow, in which parallel streamlines tend to remain parallel, with little lateral mixing, and highly chaotic "turbulent" flow, in which the fluid velocities vary in an apparently random fashion in space and time.

- We know from the study of terrestrial fluids that the transition from laminar flow to turbulent flow is strongly associated with the Reynolds number of the flow. Flows with Reynolds numbers of a few thousand or less typically remain laminar, as viscosity is able to damp out any small turbulent eddies before they can fully develop. On the other hand, flows with Reynolds numbers $\mathrm{Re} \sim 5000$ or more are frequently turbulent.

- As we discussed already in the first lecture in this course, the Reynolds number of most astrophysical flows is very large, $\mathrm{Re} \gg 5000$. We would therefore expect turbulence to be a common feature of many (if not all) astrophysical flows.

- This presents us with a big problem: how do we model the behaviour of a flow in which the velocity is varying randomly?

- If the behaviour of the flow was truly random, we would have no chance of developing any kind of predictive theory for its behaviour. However, we know from the study of other non-linear, chaotic systems that their behaviour is not truly random. There are underlying regularities in their behaviour that we might hope to understand with the aid of a suitable statistical theory. Therefore, even if the motion of the fluid on very small scales is chaotic, and hence to all intents and purposes non-deterministic, we can still try to develop a predictive statistical theory.

- Unfortunately, doing so has proved to be an extremely hard problem. Although many features of turbulent flows are now understood to some extent, no complete analytically theory exists that fully describes the statistical behaviour of turbulent flows. In addition, despite valiant efforts, numerical models remain a relative crude tool for modelling turbulence, since small-scale numerical viscosity limits the effective Reynolds number of the model flows to no more than a few thousand, even in the highest resolution simulations.

- The goal of this pair of lectures is to give you a brief introduction to the fascinating and complicated problem of turbulence. We will start by examining the simplest possible case – unmagnetized, incompressible turbulence – before turning our attention to the type of turbulence that is more representative of real astrophysical flows, which is both supersonic (and hence highly compressible) and magnetized.

6.1 Incompressible turbulence

6.1.1 Correlations functions and the kinematics of turbulence

- In order to characterize turbulence in a statistical fashion, we need to work in terms of average properties. However, as turbulent flows are highly variable in both space and time, it is not immediately obvious whether we should work in terms of spatial averages or time averages.

- Since there seems no reason *a priori* to prefer one form of averaging over the other, we generally assume that the system is **ergodic**, meaning that the two averages are equivalent.

- Another common simplifying assumption is that turbulence is already **fully developed** – i.e. that the flow is not undergoing a transition from smooth to turbulent flow, but instead is already fully turbulent.

- Finally, we typically also assume that the turbulence is **homogeneous** and **isotropic**, i.e. without a preferred location or direction. Note that this does not mean that the flow as a whole has no preferred direction, since way may often be in the situation of having turbulent fluctuations around some underlying bulk flow. In this case we have at any point in the flow
$$\vec{v}(\vec{x}) = \vec{v}_{\rm mean}(\vec{x}) + \vec{v}_{\rm turb}(\vec{x}), \tag{618}$$
where $\bar{\vec{v}}_{\rm turb} = 0$. However, since we can always choose to work in a frame moving with the flow, we will assume for simplicity that we can set $\vec{v}_{\rm mean} = 0$.

- One of the main tools that we can use for quantifying the properties of a turbulent flow is the **velocity correlation tensor**
$$\overline{\vec{v}(\vec{x}, t)\vec{v}(\vec{x} + \vec{r}, t)}.$$
(also referred to as the velocity correlation function). Here, the vector $\vec{r}$ is also known as the **lag**. When $\vec{r} = 0$, this simply measures the average value of the square of the velocity, and hence is a measure of the kinetic energy of the flow. However, when $\vec{r}$ is non-zero, the value of this expression is a measure of how well correlated the velocities are at different points in the flow.

- We expect that as $r \equiv |\vec{r}| \rightarrow \infty$, the value of the velocity correlation tensor should drop to zero, i.e.
$$\lim_{r \rightarrow \infty} \overline{\vec{v}(\vec{x}, t)\vec{v}(\vec{x} + \vec{r}, t)} = 0. \tag{619}$$
The maximum value of r for which the tensor has a substantially non-zero value is known as the **correlation length** of the turbulence.

- The assumptions of homogeneity and isotropy allow us to deduce a number of properties of the velocity correlation tensor using symmetry arguments. For example, homogeneity implies that $\overline{v_i(\vec{x})v_j(\vec{x} + \vec{r})}$ is independent of $\vec{x}$, and isotropy implies that it must also be independent of the direction of $\vec{r}$. Therefore, we can write the components of the tensor as

$$R_{ij}(r) = \overline{v_i(\vec{x})v_j(\vec{x}+\vec{r})} \tag{620}$$

- From this, it follows that

$$\frac{\partial R_{ij}}{\partial r_j} = \overline{v_i(\vec{x})\frac{\partial v_j(\vec{x}+\vec{r})}{\partial r_j}}. \tag{621}$$

However, for an incompressible fluid, $\nabla \cdot \vec{v} = 0$, and hence

$$\frac{\partial R_{ij}}{\partial r_j} = 0. \tag{622}$$

Similarly, by symmetry we also have

$$\frac{\partial R_{ij}}{\partial r_i} = 0. \tag{623}$$

- Isotropy also tells us that $R_{ij}(\vec{r}) = R_{ij}(-\vec{r})$, and hence R_{ij} can depend only on even powers of r. We can therefore write it as

$$R_{ij}(r) = A(r)r_i r_j + B(r)\delta_{ij}, \tag{624}$$

where $A(r)$ and $B(r)$ are scalar functions of r.

- Let us now decompose the velocity correlation function into two components, one longitudinal (i.e. along the flow) and one lateral (i.e. perpendicular to the flow). The longitudinal component of $\vec{r}$ is simply r, and hence

$$R_{ll}(r) = A(r)r^2 + B(r) = \frac{1}{3}\overline{v^2}f(r), \tag{625}$$

where $f(r)$ has been normalized so that $f(0) = 1$. For the lateral correlation function we have instead

$$R_{nn}(r) = B(r) = \frac{1}{3}\overline{v^2}g(r), \tag{626}$$

since $r_n = 0$. Note that again we have normalized the function $g(r)$ so that $g(0) = 1$. Therefore, we can write the correlation tensor in terms of $f(r)$ and $g(r)$ as

$$R_{ij} = \frac{1}{3}\overline{v^2}\left[\frac{f(r) - g(r)}{r^2} r_i r_j + g(r)\delta_{ij}\right]. \tag{627}$$

[You can easily verify that this expression reduces to the correct expressions for R_{ll} and R_{nn}, and also yields $R_{ln} = R_{nl} = 0$, in agreement with Equation 624].

- Note that the functions $f(r)$ and $g(r)$ here are not independent: the fact that R_{ij} must satisfy the incompressibility condition (Equation 622) tells us that $f(r)$ and $g(r)$ are linked, and indeed it is possible to completely determine one given the other. We therefore need constrain only a single scalar function in real space in order to completely determine the velocity correlation tensor $R_{ij}(r)$.

- Similarly, if we consider the Fourier transform of $R_{ij}(r)$

$$\Phi_{ij}(k) = \frac{1}{(2\pi)^3} \int R_{ij}(r) e^{i\vec{k}\cdot\vec{r}} \mathrm{d}^3 r, \tag{628}$$

then incompressibility implies that

$$k_i \Phi_{ij} = k_j \Phi_{ij} = 0, \tag{629}$$

and isotropy implies that

$$\Phi_{ij}(k) = C(k) k_i k_j + D(k) \delta_{ij}. \tag{630}$$

- As before, the incompressibility condition implies that $C(k)$ and $D(k)$ are not independent, and that we can write Φ_{ij} in terms of a single scalar function $E(k)$:

$$\Phi_{ij}(k) = \frac{E(k)}{4\pi k^4} \left(k^2 \delta_{ij} - k_i k_j \right). \tag{631}$$

- The kinetic energy density of our turbulent fluid is given by

$$\frac{1}{2}\overline{v^2} = \frac{1}{2} R_{ii}(0) = \frac{1}{2} \int \Phi_{ii}(k) \mathrm{d}^3 k. \tag{632}$$

Rewriting this in terms of $E(k)$, we find that

$$\frac{1}{2}\overline{v^2} = \int_0^\infty E(k) \mathrm{d}k. \tag{633}$$

- We therefore see that $E(k)$ is the **energy spectrum** of our turbulence, and that in the special case of homogeneous, isotropic, incompressible turbulence, knowledge of $E(k)$ alone is sufficient to determine the velocity correlation tensor for all lengths r.

- This is as far as symmetry considerations alone can take us. In order to proceed further, we need to derive an expression for $E(k)$. Before we do this, however, we first take a minor detour and discuss the issue of **vorticity**.

6.1.2 Vorticity, vortex tubes and turbulent eddies

- Consider the Euler equation for a fluid in the absence of gravitational effects:

$$\frac{\partial \vec{v}}{\partial t} + (\vec{v} \cdot \nabla) \vec{v} = -\frac{1}{\rho} \nabla p. \tag{634}$$

For any vector field $\vec{v}$, the following identity holds:

$$(\vec{v} \cdot \nabla) \vec{v} = \frac{1}{2} \nabla (\vec{v} \cdot \vec{v}) - \vec{v} \times (\nabla \times \vec{v}). \tag{635}$$

We can therefore write the Euler equation in the mathematically equivalent form:

$$\frac{\partial \vec{v}}{\partial t} + \frac{1}{2} \nabla (\vec{v} \cdot \vec{v}) - \vec{v} \times (\nabla \times \vec{v}) = -\frac{1}{\rho} \nabla p. \tag{636}$$

- If we now take the curl of this expression, and introduce the **vorticity**

$$\omega \equiv \nabla \times \vec{v}, \tag{637}$$

then we find that

$$\frac{\partial \omega}{\partial t} = \nabla \times (\vec{v} \times \omega) + \frac{1}{\rho^2} \nabla \rho \times \nabla p. \tag{638}$$

- For an incompressible fluid, $\nabla \rho = \nabla p = 0$, and this simplifies to

$$\frac{\partial \omega}{\partial t} = \nabla \times (\vec{v} \times \omega). \tag{639}$$

- The form of this equation should be familiar to you – it is the same as that of the induction equation for the magnetic field in the case of ideal MHD. We have already seen that this form of the induction equation implies flux freezing, i.e. that the amount of magnetic flux passing through a given surface S is invariant with time:

$$\frac{\mathrm{d}}{\mathrm{d}t} \int_S \vec{B} \cdot \mathrm{d}\vec{A} = 0. \tag{640}$$

Similarly, the fact that we can write the vorticity equation in this form implies that a similar result holds for the vorticity:

$$\frac{\mathrm{d}}{\mathrm{d}t} \int_S \omega \cdot \mathrm{d}\vec{A} = 0. \tag{641}$$

This is **Kelvin's vorticity theorem**.

- What is the relevance of this result for our study of turbulence? For any velocity field, we can always perform what is known as a **Helmholz decomposition**, writing it as the sum of a solenoidal (i.e. zero divergence) and an irrotational (i.e. zero curl) component. In an incompressible fluid, the irrotational component, which has non-zero divergence, is absent by definition, and the velocity field is purely solenoidal, with $\nabla \cdot \vec{v} = 0$ and $\nabla \times \vec{v} \neq 0$.

- Our turbulent, incompressible fluid therefore always has a non-zero vorticity, and we can picture the velocity field as being made up of many **eddies** with different sizes.

- Now consider an eddy threaded by a line of constant ω – sometimes known as a **vortex tube**. Let P and Q be two fluid elements located at different positions along this vortex tube. Owing to the turbulence, P and Q will move randomly and will therefore tend to move apart (since there are many more possible configurations of the fluid where they are far apart then there are where they are close together). Kelvin's theorem tells us that the vorticity carried by the vortex tube threading through these two fluid elements is conserved, and so the vortex tube will become stretched. However, if the fluid is incompressible, the volume of the vortex tube must be conserved, so as it stretches, its cross-section must decrease, corresponding to the eddy shrinking.

- We therefore see that, in general, large turbulent eddies tend to become smaller turbulent eddies – in other words, there is a cascade of energy from large scales to smaller scales within the flow. This observation forms the basis for Kolmogorov's theory of incompressible turbulence, which we examine in the next section.

6.1.3 Kolmogorov's theory

- Consider a turbulent, incompressible fluid. Energy is fed into this fluid at a rate ϵ per unit mass per unit time into eddies with size L and characteristic velocity V.

- The Reynolds number of these eddies is approximately
$$\mathrm{Re} \sim \frac{LV}{\nu}, \tag{642}$$
where ν is the kinematic viscosity. In order for the fluid to be turbulence, we must have $\mathrm{Re} \gg 1$, and so the effects of viscous dissipation on this scale are unimportant.

- Now let us assume that energy cascades from these large eddies down to smaller scales, due to processes like the vortex stretching described in the previous section. We also assume that the turbulence is in a steady state, so that energy does not accumulate at any point in the flow. This then implies that the rate at which energy is transferred from large eddies to smaller eddies must be ϵ, *independent of the size of the eddies*.

- Let us now suppose that eddies with size l have some characteristic velocity v associated with them. We make the reasonable assumption that it is possible to write the energy flow rate ϵ in terms of v and l. Then, it follows from dimensional analysis that
$$\epsilon \sim \frac{v^3}{l}, \tag{643}$$
from which it follows that
$$v \sim (\epsilon l)^{1/3}. \tag{644}$$

- We therefore see that once we fix ϵ, there is a power-law relationship between the velocity of the eddy and its size, with smaller eddies having lower associated velocities than larger eddies.

- The Reynolds number associated with an eddy of size l is simply $\mathrm{Re} \sim vl/\nu \propto l^{4/3}/\nu$. Therefore, as the size of the eddies decreases, the Reynolds number also decreases. Eventually, we reach a regime where $\mathrm{Re} \sim 1$, at which point the cascade of energy from larger scales to smaller scales stops, and the energy is dissipated by viscosity.

- If we write the size and velocity of these viscosity-dominated eddies as l_{d} and v_{d}, where $l_{\mathrm{d}} v_{\mathrm{d}} \sim \nu$, then it follows that
$$\epsilon \sim \frac{v_{\mathrm{d}}^3}{l_{\mathrm{d}}}. \tag{645}$$
Therefore,
$$l_{\mathrm{d}} = \left(\frac{\nu^3}{\epsilon}\right)^{1/4}, \qquad v_{\mathrm{d}} \sim (\nu\epsilon)^{1/4}. \tag{646}$$
Moreover, since $\epsilon \sim V^3/L$, it follows that
$$\frac{L}{l_{\mathrm{d}}} \sim \mathrm{Re}^{3/4}, \qquad \frac{V}{v_{\mathrm{d}}} \sim \mathrm{Re}^{1/4}. \tag{647}$$

Therefore, the size of the smallest eddies and the velocity associated with them is determined only by the size and velocity of the largest eddies and the Reynolds number associated with them.

- We now consider what these results imply for the energy spectrum $E(k)$. If the largest eddies have a physical size L, then this implies that $E(k)$ will drop rapidly to zero for wavenumbers smaller than $k_{\rm L} \sim 1/L$. If the size of the smallest eddies is $l_{\rm d}$, then we also expect there to be a cutoff in $E(k)$ for $k > k_{\rm d} \sim 1/l_{\rm d}$.

- The range of wavenumbers between these two limits, $k_{\rm L} < k < k_{\rm d}$, is known as the **inertial range**. Within the inertial range, energy is cascading steadily from larger scales to smaller scales, and we expect that in this regime, $E(k)$ will depend only on k and the energy flow rate ϵ. Dimensional analysis then implies that $E(k)$ in the inertial range must be of the form
$$E(k) = C\epsilon^{2/3}k^{-5/3}, \tag{648}$$
where C is a dimensionless constant.

- The fact that $E(k) \propto k^{-5/3}$ in the inertial range is one of the key results of Kolmogorov's theory, and has become known as **Kolmogorov's 5/3 law**. It is important because, as we have already seen, knowledge of $E(k)$ allows us to determine the full velocity correlation functions for isotropic, homogeneous turbulence.

- It is important to note that our "derivation" of this law rests on a large number of assumptions, some or all of which may be incorrect. However, it turns out that the results we obtain using this simple line of argument do indeed provide a good description of the behaviour of incompressible turbulence, since detailed laboratory measurements confirm the existence of an inertial range for $E(k)$ and also show that within this inertial range, $E(k) \propto k^{-5/3}$, as expected.

6.2 Supersonic turbulence

- We now turn our attention to the problem of supersonic turbulence. In many terrestrial applications, we are dealing with highly subsonic flows that are effectively incompressible, with $\nabla \cdot \vec{v}$ very small even if not exactly zero. In astrophysical fluids, however, we are often working in a regime where the flow is highly supersonic. In this regime, we cannot simply set $\nabla \cdot \vec{v} = 0$, and indeed must contend with the fact that our flow will, in general, no longer be continuous, but instead will be full of shocks.

- Treatment of turbulence in this regime is harder and less well understood than in the incompressible, subsonic regime. In this section, we will look at what we can deduce about the behaviour of $E(k)$ in this regime, and then go on to explore some of the empirical results concerning the behaviour of supersonic turbulence that come from numerical simulations of this process.

6.2.1 Burgers turbulence and other models

- One of the key points on which Kolmogorov's analysis rests is the assumption that the flow of energy from one mode to another in k space is the same at all k within the inertial range. In other words, the energy that at any given instant is carried by modes with wavenumber k_1 will at a short time in the future be carried by modes with wavenumber $k_2 > k_1$, where k_2 is only slightly larger than k_1.

- In a supersonic flow, the presence of shock waves means that this assumption is no longer valid. In a shock, energy can pass rapidly from a mode with $k = k_1$ to a mode with $k = k_3 \gg k_1$ without cascading through the modes in between. For this reason, we cannot simply apply the Kolmogorov analysis to supersonic flow.

- An alternative model for the behaviour of turbulence in supersonic flow is provided by **Burgers turbulence**. In this case, we ignore pressure forces and consider the behaviour of the simplified Navier-Stokes equation

$$\frac{\partial \vec{v}}{\partial t} + (\vec{v} \cdot \nabla)\vec{v} = \mu \nabla^2 \vec{v}. \tag{649}$$

  In one dimension, this simplifies even further to

$$\frac{\partial v}{\partial t} + v \frac{\partial v}{\partial x} = \mu \frac{\partial^2 v}{\partial x^2}. \tag{650}$$

- In general, solutions to this equation will consist of a series of uncorrelated shocks. It can be shown that in this case, the energy spectrum in the inertial range has a steeper dependence on wavenumber:

$$E(k) \propto k^{-2}. \tag{651}$$

  Outside of the inertial range, we expect that the behaviour of $E(k)$ will be very similar to that in the incompressible case: at large k, its shape will be determined by the process responsible for injecting energy into the turbulent motions, while on small scales $k \sim 1/l_{\rm shock}$, where $l_{\rm shock}$ is the thickness of a typical shock, dissipation will dominate and $E(k)$ will once again fall off rapidly.

- Detailed numerical simulations of the behaviour of supersonic turbulence confirm this basic picture for the behaviour of $E(k)$, but also show that neither the Kolmogorov nor the Burgers analysis produce the correct scaling of $E(k)$ with k. Empirically, we find that for small k within the inertial range, $E(k) \propto k^{-1.74}$ – in other words, the scaling is steeper than for incompressible turbulence, but not nearly as steep as predicted by the Burgers analysis.

6.2.2 The density and column density PDFs

- In incompressible turbulence the density field is, by definition, unaffected by the flow. In compressible turbulence, on the other hand, the turbulence flow can and does perturb the density distribution of the gas.

- In our discussion of shocks, we saw that in an adiabatic gas with $\gamma = 5/3$, the increase in the density produced by even the strongest shocks never exceeds a factor of 4. In an isothermal gas, on the other hand, the density jump in the strong shock limit scales with the Mach number as $\rho_2/\rho_1 \propto \mathcal{M}^2$. Therefore, in highly supersonic isothermal turbulence, large density changes can be produced.

- The real ISM is neither adiabatic nor isothermal, but its behaviour more closely resembles an isothermal gas than an adiabatic gas, particularly in regions such as giant molecular clouds (GMCs) where the gas temperature depends only very weakly on the density.

- It is therefore no surprise that simulations of supersonic turbulence in isothermal gas that is not self-gravitating find that a wide range of different densities is produced by the action of the turbulence. What is at first sight a little more surprising is that the density distribution resulting from the turbulence takes on a very simple form.

- Let us define a logarithmic density variable

$$s = \ln\left(\frac{\rho}{\rho_0}\right), \tag{652}$$

where ρ_0 is the mean density of the gas. Simulations show that the probability density function (PDF) of this logarithmic density is well approximated by a Gaussian distribution

$$p(s) = \frac{1}{\sqrt{2\pi\sigma_s^2}} \exp\left[-\frac{(s \pm s_0)^2}{2\sigma_s^2}\right], \tag{653}$$

where s_0 is the (volume-weighted) mean value of s, σ_s^2 is its variance, and where we take the plus sign in the exponent if we are interested in the mass-weighted PDF (i.e. the fraction of the total mass that corresponds to each small interval $s, s + \mathrm{d}s$) and the minus sign if we are interested in the volume-weighted PDF (i.e. the fraction of the total volume corresponding to each small interval in s).

- The fact that the density PDF has a log-normal form can actually be understood as a consequence of the Central Limit Theorem (CLT). Suppose that a parcel of gas with initial density $\rho(t_0)$ passes through a series of shocks and rarefactions, each of which leads to a change in its density. If we denote the density ratio produced by the i-th shock or rarefaction as δ_i, then the final density of the parcel of gas will be given by

$$\rho(t_\mathrm{f}) = \prod_i \delta_i \rho(t_0). \tag{654}$$

- If we now take the log of both sides of this expression, we find that

$$\ln\rho(t_\mathrm{f}) = \ln\rho(t_0) + \sum_i \ln\delta_i. \tag{655}$$

If the individual density changes are uncorrelated, which is a reasonable first approximation for a turbulent flow, then the right-hand side of this expression is essentially just the sum of a large number of independent random variables, and we know from the CLT that the result of this is a Gaussian distribution. Therefore, we expect the logarithm of the density to have a normal distribution, and the density its to have a **log-normal distribution**.

- The other important regularity in the details of the density PDF that has been uncovered by numerical simulations is the fact that there is a simple relationship between σ_s^2 and a small number of properties of the flow. Specifically,

$$\sigma_s^2 = \ln\left(1 + \frac{\beta}{\beta+1} b^2 \mathcal{M}^2\right), \tag{656}$$

where $\beta = p_{\rm therm}/p_{\rm mag}$ is the plasma beta parameter discussed in a previous lecture, $\mathcal{M}$ is the rms Mach number of the turbulence, and b is a function of the turbulence forcing.

- If turbulence is driven by a fully solenoidal (divergenceless) process, then $b = 1/3$, while fully compressive driving yields $b = 1$. A natural mix of modes (two-thirds solenoidal, one-third compressive) yields $b \simeq 0.4$.

- We therefore see that the more compressive the driving, the larger the range of densities that is produced. Similarly, increasing the Mach number also increases the range of densities (since it creates stronger shocks with larger density ratios). On the other hand, increasing the magnetic field strength at fixed $p_{\rm therm}$ decreases β and hence decreases the range of densities produced by the turbulence. This last result is a consequence of the fact that as the magnetic field grows stronger, magnetic pressure plays an increasingly important role in the shocks that occur, and the resulting density jumps become smaller and smaller.

- In practice, it is very difficult to measure the volume density PDF of real astrophysical systems, as we have few observational tracers that give unambiguous information on the density of the gas. Instead, most observational studies of the density distribution of interstellar clouds have focussed on the **column density PDF**. Simulations show that if the volume density PDF is log-normal, then so is the column density PDF, although the dispersion about the mean is much larger for the former than for the latter.

- This reduction in the dispersion is a simple effect of averaging along the line of sight. Suppose we split up one particular line of sight into a large number of segments of length ΔL. The column density along that line of sight can then be written as

$$\Sigma = \sum_{i=1}^{N} \rho_i \Delta L, \tag{657}$$

where ρ_i is the density associated with segment number i, and $N\Delta L = L$, the total path length. If the density in each segment is uncorrelated from the next, then we are essentially drawing N random variables from our density PDF. The mean of these N

samples, in the limit where $N \to \infty$, is simply the mean of the density PDF. In the limit $N \to \infty$, we therefore have $\Sigma = \bar{\rho}L$, and since we can use the same argument for any line of sight, we wind up with a column density PDF that is a delta function.

- In reality, of course, the densities of the individual segments are not completely uncorrelated. Nevertheless, if the size of the cloud along the line of sight is significantly larger than the typical correlation length of the density field, we can apply a similar argument. However, in this case, we cannot let N tend to infinity; instead, it is limited to $N \sim L/l_{\rm corr}$, where $l_{\rm corr}$ is the correlation length of the density field (as otherwise our assumption of random sampling of the volume density PDF breaks down). We therefore do not recover the same column density for each line of sight, but instead find values that vary around $\bar{\rho}L$ with a dispersion that is roughly

$$\sigma_{\Sigma} \sim \left(\frac{l_{\rm corr}}{L}\right)^{1/2} \sigma_{\rho}. \tag{658}$$

- All of the above results are for gas which is not self-gravitating, i.e. that nowhere has an accumulation of mass $> M_{\rm J}$. What happens if we relax this assumption?

- If we allow the gas to undergo gravitational collapse in localized regions whose masses exceed the local value of the Jeans mass, then the fact that $M_{\rm J} \propto n^{-1/2}$ means that the regions that collapse first will be the densest ones. At high gas densities, therefore, gravity takes over from turbulence as the main process responsible for structuring the gas.

- The density distribution of regions dominated by self-gravity is not log-normal. These regions tend to develop power-law radial density profiles, $\rho \propto r^{-\alpha}$. Within these regions, the probability of selecting a particular density depends on the fraction of the volume or mass that has that density (depending on whether we are computing a volume-weighted or mass-weighted PDF).

- If our collapsing region has a total mass M, an outer radius R, and a density profile $\rho = \rho_0 (R/r)^{\alpha}$, then the total mass with a density in the interval $\rho \to \rho + {\rm d}\rho$ is given by

$$\begin{aligned} {\rm d}M &= 4\pi\alpha^{-1} r^3 {\rm d}\rho, & (659) \\ &= \left(\frac{3-\alpha}{\alpha}\right) M \rho^{-3/\alpha} \rho_0^{(3-\alpha)/\alpha}. & (660) \end{aligned}$$

The mass fraction in this density interval therefore scales with the density as $f_{\rm M} \propto \rho^{-3/\alpha}$, and hence the mass-weighted density PDF in this region will have the form of a power law with index $-3/\alpha$. A similar analysis shows that the volume-weighted PDF will also have a power-law form.

- At high densities, the power-law contributions to the total density PDF made by our collapsing regions contribute more than the log-normal contribution coming from the region dominated by turbulence, even if only a small fraction of the mass is found in collapsing regions. The full volume density PDF therefore develops a power-law tail once regions of the gas start to collapse, as does the column density PDF.

- Observations of star-forming and starless clouds in the ISM provide strong support for this result. In starless clouds, the column density PDF is well-described by a log-normal function, with no high-density power-law tail. On the other hand, star-forming clouds have column density PDFs that have power law tails at high density.

6.2.3 Kinetic energy decay rates

- Another important empirical result concerns the rate at which energy is dissipated by supersonic turbulence. We can estimate this using dimensional analysis. If we define a **turbulent crossing time**
$$t_{\rm cross} = \frac{L}{\sigma}, \tag{661}$$
where L is the largest driving scale of the turbulence and σ is the three-dimensional velocity dispersion, then a plausible estimate for the kinetic energy decay rate is
$$\dot{u}_{\rm kin} \sim \frac{u_{\rm kin}}{t_{\rm cross}} = \frac{1}{2}\frac{\sigma^3}{L}, \tag{662}$$
where $u_{\rm kin}$ is the specific kinetic energy (i.e. the kinetic energy per unit mass). Simulations of driven turbulence show that this estimate is surprisingly accurate.

- If we don't drive the turbulence by continually inputting energy on large scales, but instead simply allow the initial turbulent kinetic energy to decay, then we face a couple of additional problems in defining an energy decay rate. First, the velocity dispersion σ will of course decrease with time as the kinetic energy is lost. Second, and more importantly, it is no longer obvious what scale we should take to characterize the turbulence.

- If we assume that L remains fixed as the turbulence decays, then we can write the turbulent crossing time as
$$t_{\rm cross} \sim \frac{L}{u_{\rm kin}^{1/2}}, \tag{663}$$
allowing us to derive the following estimate for the decay rate:
$$\dot{u}_{\rm kin} \sim \frac{u_{\rm kin}^{3/2}}{L}. \tag{664}$$
Solving this, we find that in this case, $u_{\rm kin}(t) \propto t^{-2}$.

- Numerical simulations of decaying turbulence do not recover this result. Instead, they find that $u_{\rm kin}(t) \propto t^{-\eta}$, with $\eta \sim 1$. The reasons for this discrepancy are not entirely understood, but may be related to the effective length scale of the turbulence. For example, if we assume that $L(t) \propto t^{1/2}$, rather than remaining fixed, then we recover the empirical relationship.

6.2.4 Turbulent heating

- Dissipation of turbulent kinetic energy in the ISM does not occur in a spatially uniform fashion. Instead, the dissipation is highly **intermittent**.

- This intermittency is one of the characteristics features of turbulent flows and is a consequence of the fact that the structures where the dissipation occurs have a lower dimensionality that the fluid as a whole. Consider, for example, the incompressible case discussed previously. Dissipation in incompressible flows occurs in eddies with widths comparable to the viscous dissipation scale. Since the length of an eddy shrinks as its width decreases, the eddies responsible for the dissipation are typically highly extended and narrow, i.e. they are basically 1D filamentary structures. We can therefore picture dissipation in a 3D incompressible turbulent flow as occurring in a tangled network of thin filaments that fill only a small fraction of the total volume of the fluid.

- In supersonic hydrodynamical turbulence, on the other hand, dissipation occurs primarily in shocks rather than in eddies. Therefore, the regions dominating the dissipation are 2D, sheet-like structures, rather than 1D filaments. However, these sheets are thin – recall that the thickness of a shock is of the order of a few particle mean-free-paths – and so the dissipative structures again fill only a small fraction of the volume.

- In supersonic MHD turbulence, things are slightly more complicated. Shocks remain important sources of dissipation, particularly for flows oriented along the field lines, but significant energy dissipation also occurs in **current sheets**. These are sheet-like regions in the flow where the magnetic field reverses direction (so that adjacent field lines are anti-parallel), created by tangling of the field by the turbulence. A field reversal implies a large local value of $\nabla \times \vec{B}$, and hence a large current flow, since

  $$\nabla \times \vec{B} = \frac{4\pi}{c} \vec{j}_{\rm e} \tag{665}$$

  in the MHD regime. The thickness of a current sheet is very small and within this region the MHD approximation can break down, with plasma physics effects such as reconnection becoming important. These can dissipate significant magnetic energy, which in turn leads to a reduction in the turbulent kinetic energy if this is in approximate equipartition with the magnetic energy.

- Why does the intermittent nature of dissipation in turbulent flows matter? From the point of view of understanding the thermal balance of the ISM, it matters because it plays an important role in determining the importance of turbulent heating.

- It's easiest to illustrate this with a simple example. Consider a molecular cloud with size 10 pc and turbulent velocity dispersion $3 {\rm km\,s^{-1}}$. The mean rate at which turbulent kinetic energy is converted to heat in this cloud is

  $$\dot{U} = \frac{\sigma^3}{L} \simeq 8.75 \times 10^{-4} \, {\rm erg\,s^{-1}\,g^{-1}}. \tag{666}$$

If we assume that dissipation of this energy is uniform, then the corresponding turbulent heating rate per unit volume is

$$\Gamma_{\rm turb} = \rho \dot{U} \simeq 6 \times 10^{-25} \left(\frac{n}{300\ {\rm cm}^{-3}}\right)\ {\rm erg\,s^{-1}\,cm^{-3}} \tag{667}$$

where n is the hydrogen nuclei number density, and a value of 300 cm^{-3} is fairly representative of local GMCs.

- For comparison, the cosmic ray heating rate per unit volume in the local ISM can be written as

$$\Gamma_{\rm cr} = 20\,{\rm eV} \times \left(\frac{\zeta_{\rm H}}{10^{-16}\ {\rm s}^{-1}}\right) \left(\frac{n}{300\ {\rm cm}^{-3}}\right)\ {\rm erg\,s^{-1}\,cm^{-3}} \tag{668}$$

$$\simeq 10^{-24}\ {\rm erg\,s^{-1}\,cm^{-3}}, \tag{669}$$

where $\zeta_{\rm H}$ is the cosmic ray ionization rate of atomic hydrogen. Similarly, the photoelectric heating rate of gas in an unshielded part of the ISM can be written as[12]:

$$\Gamma_{\rm PE} \simeq 2 \times 10^{-23} \left(\frac{n}{300\ {\rm cm}^{-3}}\right)\ {\rm erg\,s^{-1}\,cm^{-3}}. \tag{670}$$

- Globally, therefore, turbulent dissipation contributes only a few percent of the total volumetric heating rate, which might lead one to imagine that it is unimportant in the thermal balance of the ISM. However, because the turbulent dissipation is so highly intermittent, it can become the dominant *local* source of heating, even if it is unimportant globally. For example, suppose that the total fraction of the cloud volume occupied by dissipation regions is $\sim 0.1\%$. Then, within these regions, the dominant heat source (by an order of magnitude) would be turbulent dissipation, while outside of these regions, turbulence would not significantly affect the thermal balance of the gas.

- This intermittent turbulent heating has important consequences for the chemical state of the ISM, as well as for its observational properties. For example, one of the biggest puzzles in the astrochemistry of the diffuse ISM is the abundance of the CH^+ ion. This is observed to be present in quantities that cannot be explained by cold chemistry initiated by the slow radiative association reactions

$$C^+ + H \rightarrow CH^+ + \gamma, \tag{671}$$

$$C^+ + H_2 \rightarrow CH_2^+ + \gamma. \tag{672}$$

It can be produced in the reaction

$$C^+ + H_2 \rightarrow CH^+ + H, \tag{673}$$

but this reaction is endothermic by around 5000 K and hence does not proceed at typical molecular cloud temperatures. However, if turbulent dissipation results in isolated pockets of gas with very high temperatures, then CH^+ can form efficiently there, explaining its unusually high abundance.

[12] In regions where the dust extinction is significant, the photoelectric heating rate is much smaller, since it scales with the dust extinction as $\exp(-2.5 A_{\rm V})$, where $A_{\rm V}$ is the visual extinction.

- Turbulent dissipation may also be responsible for observations of small-scale fluctuations in the CO emission of diffuse gas, which are difficult to explain with UV-shielding fluctuations, and for populating highly excited levels of CO and H_2.

6.3 Turbulent support of molecular clouds

- The rapid decay of turbulent kinetic energy in molecular clouds presents us with a problem when we try to estimate the rate at which we expect stars to be forming in the Milky Way. Surveys of the molecular gas in the Milky Way show that it contains a total of around $10^9\,\mathrm{M}_\odot$ of molecular (i.e. CO-bright) gas. This gas has a characteristic density of ~ 300 cm^{-3}, and hence a gravitational free-fall time $t_{\rm ff} \sim 3\,$Myr. If we assume that the molecular gas undergoes gravitational collapse and forms stars on around this timescale, then we obtain a Galactic star formation rate $\dot{M} \sim 10^9/3\times10^6 \sim 300\,\mathrm{M}_\odot\,\mathrm{yr}^{-1}$.
- In reality, the Galactic star formation rate is around $1\mathrm{M}_\odot\,\mathrm{yr}^{-1}$ per year, so this estimate is around 300 times too big. How can we explain this discrepancy?
- One possible explanation could be that molecular clouds form stars on a timescale much longer than the free-fall timescale. If the clouds are supported by random turbulent motions, then at any given time, only a small fraction of the gas may be available for star formation, explaining why the actual star formation rate is much smaller than our estimate above. However, for this explanation to be viable, it must be necessary to retain the turbulence in the cloud for a timescale $t \gg t_{\rm ff}$. Unfortunately, our estimate of the energy decay rate above suggests that this will not be possible. Local molecular clouds follow a size-linewidth relationship (Solomon et al., 1987)

$$\sigma = 3.0 \left(\frac{L}{10\,\mathrm{pc}}\right)^{1/2} \mathrm{km\,s^{-1}}, \tag{674}$$

  and hence

$$t_{\rm cross} \simeq 3 \left(\frac{L}{10\,\mathrm{pc}}\right)^{1/2} \mathrm{Myr}. \tag{675}$$

  Therefore, we would expect the turbulent kinetic energy to decay on a timescale comparable to $t_{\rm ff}$.
- This problem with turbulent support of molecular clouds was recognized very early, but a clever way of avoiding was devised. The argument runs as follows: if molecular clouds are magnetised (which we know to be true) and if the turbulence is primarily in the form of Alfven waves, then the turbulent decay time would be much longer than the simple estimate above. The reason for this is that Alfven waves are not compressive, and hence if all of the turbulent kinetic energy were in the form of Alfven waves, none would be dissipated by shocks. Instead, dissipation would occur only via non-ideal MHD effects (e.g. Ohmic or ambipolar diffision), over a much longer timescale.

- Unfortunately, this clever idea turns out to be wrong: even if we start with all of our energy in Alfven waves, their interaction with the inhomogeneous structure of the cloud quickly generates magnetosonic waves, which do compress the gas. Numerical simulations show that, in practice, turbulent dissipation occurs at more-or-less the same rate in magnetised turbulent clouds as in the $\vec{B} = 0$ case.

- Therefore, we're lead to one of two conclusions: either the molecular clouds must be very short-lived ($t_{\rm life} \sim t_{\rm ff}$) and must have a low star formation efficiency during that lifetime; or the turbulent kinetic energy in the clouds must be regularly replenished (i.e. the turbulence must be driven). (In practice, simulations of molecular cloud formation and evolution suggest that the real answer is a bit of both...)

6.4 What drives turbulence in the ISM?

- In the previous section, we argued that if star-forming molecular clouds have lifetimes $t_{\rm life} \gg t_{\rm ff}$, then it implies that the turbulence within the clouds must be driven. However, this is not the only scale on which turbulent driving is important.

- In the Milky Way, the molecular gas in the local ISM has a scale height of approximately $L \sim 150$ pc and a velocity dispersion on large scales of approximately $10 \, {\rm km \, s^{-1}}$. We therefore expect turbulence in the molecular gas to decay on a timescale

$$t_{\rm decay, H_2} \sim \frac{150 \, {\rm pc}}{10 \, {\rm km \, s^{-1}}} \sim 15 \, {\rm Myr}. \tag{676}$$

- In the atomic gas, $L \sim 1$ kpc and $\sigma \sim 20 \, {\rm km \, s^{-1}}$, and so we have

$$t_{\rm decay, HI} \sim \frac{1000 \, {\rm pc}}{20 \, {\rm km \, s^{-1}}} \sim 50 \, {\rm Myr}. \tag{677}$$

- In both cases, these timescales are orders of magnitude shorter than the age of the Milky Way, and so the fact that we observe the ISM to be turbulent means that some process or combination of processes must be continually replenishing the turbulent kinetic energy dissipated by the gas. In this section, we will explore a few of the more important physical processes responsible for driving turbulence in the ISM.

6.4.1 Large-scale processes

Accretion

- The Milky Way is currently forming stars at a rate of around $\dot{M}_* \sim 2$–$4 \, {\rm M_\odot \, yr^{-1}}$. Since the total gas mass in the Milky Way is around $M_{\rm gas} \simeq 8 \times 10^9 \, {\rm M_\odot}$, this implies that all of the gas will have been converted into stars on a timescale of 2–4 Gyr.

- Measurements of this timescale – known as the **gas depletion timescale** – in other galaxies find similar or smaller values, suggesting that our own Galaxy is not unusual in this respect.

- Since these timescales are typically a factor of a few or more shorter than the characteristic age of most galaxies ($\sim$ 10 Gyr), we need to explain why they have not already run out of gas.

- One obvious answer, which receives considerable support from simulations of galaxy formation, is that the galaxies are continually accreting gas from the intergalactic medium (IGM) to replace the gas lost in star formation. As this gas falls onto the disk, it will drive large-scale turbulent motions. If we assume that the Galaxy is in a quasi-steady state with $\dot{M}_{\rm acc} \sim \dot{M}_*$ (the "bathtub model"), then the accretion will inject turbulent energy into the ISM at a rate per unit volume

$$\dot{\epsilon}_{\rm acc} \sim \frac{1}{2}\rho \frac{\dot{M}_*}{M_{\rm gas}} v_{\rm in}^2, \tag{678}$$

  where $v_{\rm in}$ is the infall velocity of the accreted gas and ρ is the mean density of the ISM.

- The precise value of $v_{\rm in}$ depends on the details of the accretion, but to a first approximation we can take it to be equal to the circular velocity of the galaxy. Therefore, we have

$$\dot{\epsilon}_{\rm acc} \sim 6 \times 10^{-27}\,{\rm erg\,cm^{-3}\,s^{-1}} \left(\frac{\rho}{2 \times 10^{-24}\,{\rm g\,cm^{-3}}}\right) \left(\frac{\dot{M}_*}{3\,{\rm M_\odot\,yr^{-1}}}\right) \left(\frac{v_{\rm in}}{220\,{\rm km\,s^{-1}}}\right)^2, \tag{679}$$

  where the terms in brackets are all ~ 1 for the Milky Way.

Magnetorotational instability

- The gas disk of the Milky Way is weakly magnetized and in differential rotation, precisely the conditions required in order for the magnetorotational instability to operate. The MRI has can therefore act as a source of large-scale turbulent motions.

- Detailed analysis of the non-linear development of the MRI shows that the rate per unit volume with which it converts rotational into turbulent energy can be written as

$$\dot{\epsilon}_{\rm MRI} \sim \epsilon_{\rm mag}\Omega, \tag{680}$$

  where $\epsilon_{\rm mag} = B^2/8\pi$ is the energy density of the magnetic field, and Ω is the angular velocity of the disk.

- In the local ISM, $B \sim 3\,\mu$G, and $\Omega \sim 1/(200\,{\rm Myr})$. We therefore have

$$\dot{\epsilon}_{\rm MRI} \sim 6 \times 10^{-29}\,{\rm erg\,cm^{-3}\,s^{-1}} \left(\frac{B}{3\,\mu{\rm G}}\right)^2 \left(\frac{\Omega}{(200\,{\rm Myr})^{-1}}\right). \tag{681}$$

  Comparing this with our estimate of $\dot{\epsilon}_{\rm acc}$ above, we see that for the Milky Way, the MRI makes only a minor contribution to the turbulent energy density. However, in smaller galaxies with much smaller accretion rates, the MRI contribution may be much more important.

6.4.2 Stellar feedback

Supernovae

- The supernova rate in the Milky Way is quite uncertain, but is of order one per century. If we assume that each supernova releases $E_{\rm SN} = 10^{51}$ erg of energy, and that 10% of this energy is ultimately converted into turbulence, then the total turbulent energy production rate is $\sim 3 \times 10^{40} \,{\rm erg\,s^{-1}}$.

- To convert this into a rate per unit volume, we need to divide this by an appropriate volume. If we take the whole of the star-forming portion of this disk, and approximate this as a cylinder of radius $R = 15$ kpc and height $H = 300$ pc, then we have $V = \pi H R^2 \simeq 6.2 \times 10^{66} \,{\rm cm^3}$, and hence

$$\dot{\epsilon}_{\rm SN} \simeq 5 \times 10^{-27} \,{\rm erg\,cm^{-3}\,s^{-1}}. \quad (682)$$

  Comparing this with the other estimates above, we see that supernovae provide a comparable energy input to accretion, at least in the case of the Milky Way.

- Note, however, that it is not clear that we should take the volume of the disk as the appropriate volume here. Star formation is not uniformly distributed within the disk, and so the resulting supernovae will also not be uniformly distributed. Since it seems unlikely that supernovae will drive turbulence on scales much larger than the sizes of the observed SN remnants or superbubbles, we therefore expect that the turbulent energy input will also be inhomogeneously distributed.

- We therefore expect that in regions with active star formation (e.g. the Galactic Centre, the molecular ring), $\dot{\epsilon}_{\rm SN}$ may be much larger than estimated above, while in quiescent regions such as the outer disk, it may be much smaller.

Protostellar outflows

- The final driving mechanism that we will consider here are protostellar jets and outflows. We know from both simulations and observations that there is a relationship between the protostellar accretion rate and the mass outflow rate:

$$\dot{M}_{\rm jet} = f_{\rm jet} \dot{M}_{\rm acc}, \quad (683)$$

  with $0.1 < f_{\rm jet} < 0.4$ in most systems.

- If we consider jets specifically, their velocity is very large, $v_{\rm jet} \sim 200 \,{\rm km\,s^{-1}}$. As we have already seen, they will produce strong shocks at the working surfaces where they interact with the ISM. The strong radiative cooling that occurs in this shock limits the extent to which the energy of the jet can be converted into kinetic energy of the gas.

- If we assume that the jet material conserves momentum but not energy, and that the post-shock velocity is comparable to the velocity dispersion of the ISM, σ, then the fraction of energy converted into turbulent motions is approximately

$$\xi_{\rm jet} \sim \frac{\sigma}{v_{\rm jet}} \sim 0.05, \tag{684}$$

  where we have taken $\sigma \sim 10\,{\rm km\,s^{-1}}$ as a typical value for the ISM.

- Averaged over the whole galaxy, we therefore have

$$\dot{\epsilon}_{\rm jet} = \frac{1}{2}\xi_{\rm jet} f_{\rm jet} \frac{\dot{M}_{*} v_{\rm jet}^{2}}{\pi R^{2} H}, \ \sim 5 \times 10^{-29}\,{\rm erg\,cm^{-3}\,s^{-1}}. \tag{685}$$

  On galactic scales, outflows therefore make only a minor contribution to the total turbulent energy budget. However, as in the case of supernovae, their energy input is very inhomogeneously distributed, and on small scales they can make a major contribution, particularly in star-forming clouds that have yet to form any O or B type stars.