2.4 Accretion disks

2.4.1 Governing equations – radial structure

- So far, we have considered accretion flows that, by construction, have zero net angular momentum, which has allowed us to neglect the effects of rotation as the gas flows in towards our accreting object. However, this situation is unlikely to be encountered in reality: any real astrophysical flow will have at least some angular momentum.

- Consider an initially spherical shell of gas around our mass M with a small but non-zero angular momentum. If the gas in this shell conserves angular momentum as it flows in towards M, then its rotational velocity will increase as $v_{\rm rot} \propto r^{-1}$. Inflow parallel to the axis of rotation will occur in much the same fashion as it would if the gas were not rotating, but inflow perpendicular to the axis of rotation will feel an increasing degree of rotational support, and hence will proceed more slowly. Eventually, the rotational velocity will reach the Keplerian velocity, and collapse in this direction will stop entirely. The gas will therefore settle into a flattened rotating **accretion disk**.

- This process will also occur in flows that do not conserve angular momentum, unless the process responsible for transferring angular momentum away is extremely efficient. This is difficult to arrange, particularly once the inflow onto M becomes supersonic, and so we expect accretion disks to be a common feature of accreting astrophysical systems.

- It is therefore important to understand how accretion works in a disk geometry. In our initial study of disk accretion, we will make a couple of important simplifying assumptions. We will assume that the disk is axisymmetric, and that it is thin, with a scale height $h \ll R_{\rm disk}$, the radius of the disk.

- In place of the usual 3D spatial mass density ρ, we work in terms of the mass surface density Σ, defined as
$$\Sigma(R,t) \equiv \int_{-\infty}^{\infty} \rho(R,z,t)\mathrm{d}z, \tag{153}$$
where R and z are the radial and vertical coordinates in our cylindrical coordinate system. The velocity of the flow can be written as
$$\vec{v} = v_R \vec{\hat{e}}_R + v_\phi \vec{\hat{e}}_\phi, \tag{154}$$
where $\vec{\hat{e}}_R$ and $\vec{\hat{e}}_\phi$ are unit vectors in the radial and angular directions, respectively. Our assumption of axisymmetry then implies that the radial and tangential velocities v_R and v_ϕ are functions of R but not of ϕ.

- To further simplify our treatment of the problem, we assume that the mass of the gas in the disk is much less than the mass of the central object, so that we can ignore the self-gravity of the disk, and that the motion of the gas is nearly Keplerian, so that $v_R \ll v_\phi$ and
$$v_\phi \sim \sqrt{\frac{GM}{R}}. \tag{155}$$

- Having made these assumptions, it is straightforward to write down the continuity equation in our cylindrical coordinate system:

$$\frac{\partial \Sigma}{\partial t} + \frac{1}{R}\frac{\partial}{\partial R}\left(R\Sigma v_R\right) = 0. \tag{156}$$

- The momentum equation requires a little more thought. The velocity divergence of gas in an axisymmetric disk that is rotating in a perfectly Keplerian fashion can easily be shown to be zero. In our 2D cylindrical setup, we have

$$\nabla \cdot \vec{v} = \frac{1}{R}\frac{\partial}{\partial R}\left(R v_R\right) + \frac{1}{R}\frac{\partial v_\phi}{\phi}. \tag{157}$$

If the gas flow is perfectly Keplerian, then $v_{\rm R} = 0$, $v_\phi = (GM/R)^{1/2}$, and so both terms in this expression vanish. In reality, of course, the flow will not be perfectly Keplerian (since in that case there would be no accretion), but if the deviation from Keplerian rotation is small, then the velocity divergence will also be small. We therefore expect the dominant source of viscosity to be the shear viscosity acting between fluid elements orbiting at slightly different radii within the disk.

- A convenient way to explore the effects of this shear viscosity is through its effects on the angular momentum of the rotating gas. In the absence of viscosity, angular momentum is conserved within the disk, since the only force acting is the gravity of the central object, which exerts no torque upon the gas. Therefore, in the absence of viscosity, the angular momentum per unit area of the disk obeys a simple continuity equation:

$$\frac{\partial}{\partial t}\left(\Sigma R^2 \Omega\right) + \nabla \cdot \left(\Sigma R^2 \Omega \vec{v}\right) = 0. \tag{158}$$

Axisymmetry implies that the ϕ component of the divergence term is zero, and so we can rewrite this equation as

$$\frac{\partial}{\partial t}\left(\Sigma R^2 \Omega\right) + \frac{1}{R}\frac{\partial}{\partial R}\left(\Sigma R^3 \Omega v_R\right) = 0, \tag{159}$$

or alternatively

$$R\frac{\partial}{\partial t}\left(\Sigma R^2 \Omega\right) + \frac{\partial}{\partial R}\left(\Sigma R^3 \Omega v_R\right) = 0. \tag{160}$$

- If the gas in the disk is rotating in Keplerian fashion, or indeed in any form of **differential rotation** such that the angular velocity Ω varies with radius, then the difference in angular velocities results in a viscous torque acting on the gas. If we consider two thin annuli of material on either side of some surface of constant R in the disk and with vertical thickness $\mathrm{d}z$, then it is easy to show that the outer annulus exerts a viscous force

$$F_{\rm visc} = 2\pi R \mathrm{d}z \times \mu R \frac{\mathrm{d}\Omega}{\mathrm{d}R} \tag{161}$$

Writing this in terms of the kinematic viscosity ν and integrating in the vertical direction yields

$$F_{\rm visc} = 2\pi \nu \Sigma R^2 \frac{\mathrm{d}\Omega}{\mathrm{d}R}. \tag{162}$$

Finally, as this force is acting at a distance R from the centre of rotation, it is easy to see that it will exert a torque

$$T = 2\pi\nu\Sigma R^3 \frac{\mathrm{d}\Omega}{\mathrm{d}R}. \tag{163}$$

Note that if $\Omega(R)$ decreases with radius, as in a Keplerian disk, then T is negative – the viscosity acts to slow down the rotation of the inner parts of the disk and speed up the rotation of the outer parts, resulting in a net outward transport of angular momentum.

- The net torque acting on an annulus with radius R and width $\mathrm{d}R$ is the difference between the torque at the inner and outer edges of the annulus, which we will denote as $\mathrm{d}T$. The surface area of the annulus is $2\pi R\mathrm{d}R$, and so the net torque per unit area of the annulus is therefore

$$\frac{\mathrm{d}T}{2\pi R\mathrm{d}R} = \frac{1}{2\pi R}\frac{\partial T}{\partial R}. \tag{164}$$

- With the effects of this viscous torque included, our equation for the evolution of the angular momentum per unit area of the disk becomes

$$R\frac{\partial}{\partial t}\left(\Sigma R^2\Omega\right) + \frac{\partial}{\partial R}\left(\Sigma u_R R^3\Omega\right) = \frac{1}{2\pi}\frac{\partial T}{\partial R}. \tag{165}$$

Note that a similar expression can be derived directly from the Navier-Stokes equation for the gas in the disk. However, the derivation is complicated by the fact that we cannot necessarily assume that the viscosity in the disk is independent of radius.

- If we assume that Ω does not vary significantly with time (which is a valid approximation provided that the accretion time $t_{\mathrm{acc}} = M/\dot{M}$ is much longer than the orbital period of the disk), then we can combine our mass conservation and angular momentum conservation equations (Eqs. 156 and 165, respectively) to yield the following expression:

$$R\frac{\partial\Sigma}{\partial t} = -\frac{1}{2\pi}\frac{\partial}{\partial R}\left[\left\{\frac{\partial}{\partial R}\left(R^2\Omega\right)\right\}^{-1}\frac{\partial T}{\partial R}\right]. \tag{166}$$

If the rotational velocity of the disk is given by the Keplerian value

$$\Omega = \left(\frac{GM}{R^3}\right)^{1/2}, \tag{167}$$

then it follows that

$$\frac{\mathrm{d}\Omega}{\mathrm{d}R} = -\frac{3}{2}\frac{\Omega}{R}, \tag{168}$$

$$T = -3\pi\nu\Sigma R^2\Omega, \tag{169}$$

$$\frac{\partial T}{\partial R} = -3\pi R^{3/2}\Omega\frac{\partial}{\partial R}\left(\nu\Sigma R^{1/2}\right). \tag{170}$$

- We can therefore write the time evolution of the disk surface density as

$$\frac{\partial \Sigma}{\partial t} = \frac{3}{R} \frac{\partial}{\partial R} \left[R^{1/2} \frac{\partial}{\partial R} \left(\nu \Sigma R^{1/2} \right) \right]. \tag{171}$$

Given this, the radial velocity then follows straightforwardly from the equation of mass conservation:

$$v_R = -\frac{3}{R^{1/2}\Sigma} \frac{\partial}{\partial R} \left(\nu \Sigma R^{1/2} \right). \tag{172}$$

Finally, the mass flux through the disk at a radius R is simply

$$\dot{M}(R) = 2\pi R \Sigma |v_R|, \tag{173}$$

$$= 6\pi R^{1/2} \frac{\partial}{\partial R} \left(\nu \Sigma R^{1/2} \right). \tag{174}$$

- Comparing our expression for the mass flux with that for the evolution of the surface density, we see that

$$\frac{\partial \Sigma}{\partial t} = \frac{1}{2\pi R} \frac{\partial \dot{M}}{\partial R}. \tag{175}$$

We therefore see that if our accretion disk is in a steady state, with $\partial \Sigma / \partial t = 0$, then the mass flux through the disk is the same at all radii.

2.4.2 Governing equations – vertical structure

- It is common to assume that the gas in the disk is in hydrostatic equilibrium in the vertical direction. This is a reasonable assumption if the timescale to come to equilibrium – typically, the dynamical timescale of the gas – is much shorter than the time that it takes for gas to flow through the disk and onto the central object.

- In hydrostatic equilibrium, we know that

$$\frac{\mathrm{d}\Phi}{\mathrm{d}z} = -\frac{1}{\rho} \frac{\mathrm{d}p}{\mathrm{d}z} = c_{\mathrm{s}}^2 \frac{\mathrm{d}\rho}{\mathrm{d}z}. \tag{176}$$

- We have already assumed that the gravitational potential of the system is dominated by the central object. If we write the distance to object this from a point (R, z) in the disk as $r = \sqrt{R^2 + z^2}$, then it is easy to see that

$$\frac{\mathrm{d}\Phi}{\mathrm{d}z} = \frac{\mathrm{d}\Phi}{\mathrm{d}r} \frac{\mathrm{d}r}{\mathrm{d}z} = \frac{\mathrm{d}\Phi}{\mathrm{d}r} \frac{z}{r}. \tag{177}$$

The circular velocity for a test mass orbiting the central object at a distance r is defined as

$$v_{\mathrm{c}}^2 = r \frac{\mathrm{d}\Phi}{\mathrm{d}r}, \tag{178}$$

and so we can write the vertical component of the gravitational force as

$$\frac{\mathrm{d}\Phi}{\mathrm{d}z} = \frac{v_{\mathrm{c}}^2 z}{r^2}. \tag{179}$$

- If we now assume that the disk is thin, so that $r \sim R$ and $v_{\rm c}(r) \sim v_{\rm c}(R)$, we can write this as

$$\frac{{\rm d}\Phi}{{\rm d}z} = \frac{v_{\rm c}^2 z}{R^2}. \quad (180)$$

Substituting this into our equation of hydrostatic balance, we find that

$$\frac{{\rm d}\rho}{\rho} = -\frac{v_{\rm c}^2}{R^2 c_{\rm s}^2} z {\rm d}z. \quad (181)$$

This has a solution of the form

$$\rho \propto \exp\left(-\frac{z^2}{2h^2}\right), \quad (182)$$

where the scale height h is approximately

$$\frac{h}{R} \sim \frac{c_{\rm s}}{v_{\rm c}}. \quad (183)$$

Therefore, provided that the disk is cold, and that its rotation is supersonic, our assumption that it is thin is well-justified, since in this case we will have $h \ll R$. Note also that for a disk in approximately Keplerian rotation, $\Omega \simeq v_{\rm c}/R$, and so $c_{\rm s} \simeq h\Omega$.

2.4.3 The need for an anomalous viscosity

- We can use the equations derived in the previous section to estimate the typical rate at which mass will flow through a Keplerian accretion disk. From Equation 172, we see that to within an order of magnitude,

$$v_R \sim \frac{\nu}{R}. \quad (184)$$

The typical accretion timescale is therefore

$$t_\nu \sim \frac{R}{v_R} \sim \frac{R^2}{\nu}, \quad (185)$$

and the mass flow rate is

$$\dot{M} \sim 2\pi\Sigma\nu. \quad (186)$$

- If the only source of viscosity in the disk is the standard molecular viscosity, then $\nu \sim v_{\rm T}\lambda$, where $v_{\rm T}$ is the thermal velocity and λ is the mean free path. In this case, we see immediately that the inward velocity of the flow will be very small,

$$v_R \sim v_{\rm T}\frac{\lambda}{R}, \quad (187)$$

as we know that for astrophysical fluids, we are almost always in the regime where $\lambda \ll R$.

- Consider a protostellar accretion disk with a midplane temperature $T \sim 1000$ K, a midplane density $n \sim 10^{14}\,\mathrm{cm}^{-3}$, and a radius of 10 AU. In these conditions, we have $\lambda \sim 10$ cm, $v_{\mathrm{T}} \sim 2 \times 10^{5}\,\mathrm{cm\,s^{-1}}$, and hence $\nu \sim 2 \times 10^{6}\,\mathrm{cm^{2}\,s^{-1}}$. This implies that the accretion timescale in such a disk will be $t_{\nu} \sim 10^{22}$ s $\sim 10^{15}$ yr if molecular viscosity is the only source of viscosity acting to drive the inflow. This is vastly longer than the lifetime of a protostellar accretion disk, and hence implies that in a disk governed solely by molecular viscosity, little or no accretion will actually occur.

- A similar analysis applied to other accreting systems (e.g. X-ray binaries) gives similarly long accretion timescales, and associated accretion rates that are much smaller than those that are inferred from observations. We therefore see that there must be some additional "anomalous" viscosity present in the disks, with a magnitude that is much larger than the molecular viscosity, in order to explain the observed accretion rates.

2.4.4 The α-disk prescription

- The question of what physical process or processes are responsible for the anomalous viscosity in accretion disks is a difficult one to answer, and the issue is not yet entirely settled. However, it was realised by Shakura & Sunyaev in the mid-1970s that we can understand a great deal about the behaviour of real astrophysical accretion disks *without* understanding the source of the viscosity.

- If the source of the anomalous viscosity is some form of turbulent process, as seems likely given the very high Reynolds number of the flow, then we expect on dimensional grounds that $\nu \sim v_{\mathrm{turb}} L_{\mathrm{turb}}$, where v_{turb} is the characteristic turbulent velocity and L_{turb} is the size of the largest turbulent eddy. If the turbulence was highly supersonic, then it would rapidly dissipate energy in shocks, and so in practice, we expect that $v_{\mathrm{turb}} \sim c_{\mathrm{s}}$. Furthermore, if the turbulence is approximately isotropic, then the size of the largest eddies will not exceed the scale height of the disk, i.e. $L_{\mathrm{turb}} \sim h$.

- These considerations led Shakura & Sunyaev's to propose the following form for the kinematic viscosity:
$$\nu = \alpha h c_{\mathrm{s}}. \tag{188}$$
α here is a measure of our uncertainty – we expect on general physical grounds that $\alpha \leq 1$, but it could in principle by very much smaller. Nevertheless, this **α-disk** prescription allows one to solve for the physical structure of the accretion disk, the mass flow rate, etc. in terms of only this single unknown parameter.

- For a thin disk, we have already seen that
$$\frac{h}{R} \sim \frac{c_{\mathrm{s}}}{v_{\mathrm{c}}}, \tag{189}$$
and so we can also write our prescription for ν as
$$\nu = \alpha c_{\mathrm{s}}^{2} \frac{R}{v_{\mathrm{c}}}, \tag{190}$$
$$= \alpha c_{\mathrm{s}}^{2} \Omega^{-1}, \tag{191}$$

where the second line follows if we assume that the rotation is Keplerian (or approximately so).

- Substituting this value for ν into our expression for $\dot{M}$, we find that

$$\dot{M} = 2\pi\Sigma\alpha c_{\rm s}^2\Omega^{-1}. \tag{192}$$

For a Keplerian disk, $\Omega \propto R^{-3/2}$, and so we see that

$$\dot{M} \propto \Sigma\alpha c_{\rm s}^2 R^{3/2}. \tag{193}$$

If the accretion flow through the disk is steady, then $\dot{M}$ must be the same at any radius, and so we therefore expect that

$$\Sigma\alpha c_{\rm s}^2 R^{3/2} = \text{constant}. \tag{194}$$

In the simple case where α does not vary strongly with radius, then we can simplify this expression further, to

$$\Sigma T \propto R^{-3/2}, \tag{195}$$

where we have made use of the fact that $c_{\rm s}^2 \propto T$.

- To make further progress, we need to determine how the disk temperature T varies as a function of radius. Solving for T in the general case is a complicated problem, as realistic accretion disks are often optically thick over a wide range of radii. However, there are a few simple cases that we can look at that are nevertheless informative.

- Let us start by considering the case of an isothermal disk. This is a reasonable approximation if the cooling rate of the disk is a very steep function of temperature, as can happen in disks where the cooling is dominated by H^- opacity or by H_2 cooling (e.g. in population III protostellar accretion disks). In this case, Equation 195 shows us that

$$\Sigma \propto R^{-3/2}. \tag{196}$$

Moreover, since

$$h \propto \frac{c_{\rm s}}{\Omega} \propto T^{1/2} R^{3/2}, \tag{197}$$

we see that an isothermal α-disk flares strongly, and that its mean density

$$\bar{\rho} = \frac{\Sigma}{h} \tag{198}$$

falls off very steeply with radius, as $\bar{\rho} \propto R^{-3}$.

- Another simple case is an optically thin disk with an opacity κ that does not depend on temperature, whose heating is dominated by the radiation from the central accreting object. In this case, the heating rate of the disk varies with radius as

$$\Gamma \propto R^{-2}. \tag{199}$$

If the disk is in equilibrium, the cooling rate must vary with radius in the same fashion. However, we also know that the cooling rate will vary with the temperature as $\Lambda \propto T^4$. If we assume the density dependence of the heating and cooling rates is the same, it then follows from the fact that $\Gamma = \Lambda$ that

$$T^4 \propto R^{-2}, \tag{200}$$
$$T \propto R^{-1/2}. \tag{201}$$

From this, it then also follows that

$$\Sigma \propto R^{-1}, \tag{202}$$
$$h \propto R^{5/4}, \tag{203}$$
$$\bar{\rho} \propto R^{-9/4}. \tag{204}$$

- Finally, consider the case where the dissipation of energy by viscosity in the disk is the main heating source. In this case, one can show that[4]

$$T \propto R^{-3/4}, \tag{205}$$

and hence we find that

$$\Sigma \propto R^{-3/4}, \tag{206}$$
$$h \propto R^{9/8}, \tag{207}$$
$$\bar{\rho} \propto R^{-15/8}. \tag{208}$$

- These few simple examples barely scratch the surface of the subject, but show how powerful the α-disk prescription can be. This deceptively simple model for the disk viscosity, combined with a thermal model for the gas, allows us to solve for the structure of the disk in a wide variety of different physical scenarios. It is therefore not surprising that the original paper by Shakura & Sunyaev has become one of the most highly cited papers in the history of astrophysics.

- Nevertheless, the α-disk prescription still presents us with one major issue: what physical process is responsible for the anomalous viscosity? What sets the value of α? In the next section, we will examine one important source of disk viscosity that appears to be responsible for determining α in a wide variety of different types of disk: the **magnetorotational instability**.

2.4.5 Magnetorotational instability

- The magnetorotational instability (or MRI) is an instability that arises due to the action of a weak magnetic field in a differentially rotating disk. It grows rapidly, on the orbital timescale, and once it becomes non-linear can drive turbulence in the disk. The resulting **turbulent viscosity** is a leading candidate for the anomalous viscosity required in order to explain the large accretion rates inferred for most real accretion disks.

[4] We will explore this in more detail in exercise sheet 4

- The MRI was originally analyzed by Chandrasekhar in the 50's, but was then neglected and forgotten for many years, before being rediscovered by Balbus and Hawley in 1991.

- In Balbus and Hawley's seminal 1991 paper[5], they studied the MRI by starting with the fluid equations for a magnetized, differentially rotating disk and applying the tools of first-order perturbation theory. However, the resulting derivation of the conditions for the instability is rather lengthy, and so for our purposes, we will look instead at a much simpler but more physically intuitive model for how the instability operates.

- Consider two neighbouring fluid elements, A and B, rotating around a mass M at slightly different radii, r_A and r_B, in a differentially rotating disk. Suppose that the disk is magnetised and that the two fluid elements are linked by a magnetic field line that is oriented primarily vertically in the disk.

- As the disk rotates, the two fluid elements will begin to move apart, owing to the differential rotation. If the rotational velocity decreases with increasing radius in the disk, i.e. if
$$\frac{\mathrm{d}\Omega}{\mathrm{d}r} < 0, \tag{209}$$
then the inner fluid element will rotate faster than the outer fluid element, and hence will move ahead of it.

- Since the two fluid elements are linked by a magnetic field line, both will feel a magnetic tension force as the fluid elements move apart. This force will act to slow down the inner element A and speed up the outer element B. Consequently, both fluid elements will change their angular momentum: A will lose angular momentum and B will gain angular momentum.

- The response of the fluid elements to a change in angular momentum depends on how the specific angular momentum varies with radius. If it increases with increasing radius, then when fluid element A loses angular momentum, it will move inwards, to a smaller orbit. Similarly, fluid element B's gain in angular momentum will cause it to move outwards, to a larger orbit. However, this motion will **increase** the distance between the fluid elements and hence **increase** the magnetic tension force. It is therefore easy to see that the situation is unstable: any initially tiny difference in r_A and r_B will rapidly amplify, until the simple picture sketched above breaks down.

- If the specific angular momentum of the disk *decreases* with increasing radius, then the MRI does not operate. However, in this case the disk will violate the **Rayleigh stability criterion**,
$$\frac{\mathrm{d}(r^2\Omega)}{\mathrm{d}r} > 0, \tag{210}$$
and will be unstable to purely hydrodynamical perturbations.

[5] "A powerful local shear instability in weakly magnetized disks. I - Linear analysis.", Balbus, S. A., Hawley, J. F., 1991, ApJ, 376, 214

- A more detailed analysis of the instability shows that in practice, the disk will be susceptible to the MRI only if the magnetic field is "weak" in the sense that

$$\frac{B_z^2}{8\pi} < \frac{3}{\pi^2}\rho c_{\mathrm{s}}^2, \tag{211}$$

where B_z is the vertical component of the magnetic field, and only if the rotational velocity of the disk satisfies

$$\frac{\mathrm{d}}{\mathrm{d}r}\left(\Omega^2\right) < 0. \tag{212}$$

- Keplerian accretion disks always satisfy the second of these criteria, and hence are MRI-unstable whenever the vertical field threading them is sufficiently weak. Note that if the field is too strong, the magnetic tension between our hypothetical fluid elements A and B is strong enough to prevent them from separating significantly, thereby preventing the instability from developing.

- Although a simple linear analysis of the MRI breaks down once the perturbations become large, numerical simulations show that the non-linear evolution of the instability rapidly leads to turbulence. The MRI is therefore a simple but effective way to drive turbulence in a magnetised accretion disk.

- How do we then get from this to a prescription for the anomalous viscosity? For our α-disk model, the key feature of the viscosity is that it allows for the radial transport of angular momentum through the disk. Random "turbulent" motions can also lead to such an outward transport of angular momentum, by a process sometimes known as turbulent viscosity, but which is better thought of as the **turbulent diffusion** of momentum.

- The characteristic length scale over which molecular viscosity transfers momentum is the particle mean free path, λ. In contrast, the characteristic length scale over which turbulent viscosity transfers momentum is the scale of the largest turbulent eddy, L_{turb}. Since it is often the case that $L_{\mathrm{turb}} \gg \lambda$, turbulent viscosity provides a much more effective means of transporting angular momentum through the disk than molecular viscosity.

- Numerical models of MRI-driven turbulence in accretion disks show that the resulting large-scale behaviour of the disk is reasonably well described by the α-disk model, with $\alpha \sim 0.01$. Recalling our discussion of the disk accretion timescale, we find that in this case

$$t_\nu \sim \frac{R^2}{\alpha h c_{\mathrm{s}}} \sim \frac{R^2\Omega}{\alpha c_{\mathrm{s}}^2}. \tag{213}$$

Substituting in some plausible numbers for a protostellar accretion disk – $T \sim 1000\,\mathrm{K}$, $R \sim 10\,\mathrm{AU}$, $\Omega \sim 0.2\,\mathrm{yr}^{-1}$ – we find that $t_\nu \sim 4\times10^9\alpha^{-1} \sim 4\times10^{11}$ s for $\alpha = 0.01$, i.e. a timescale of around 10^4 yr. This is much more consistent with observations that the timescale of 10^{15} yr that we obtained when we only considered molecular viscosity.