Astrophysical Fluid Dynamics

Assignment #1: due May 2nd

1 Validity of the fluid approximation

Astrophysical fluids, such as the gas in the interstellar medium (ISM) or the plasma in stars, are ultimately made of particles. When does it make sense to approximate this collection of particles as a continuous fluid? We have seen in the lectures that the fluid approximation is valid if the mean free path of a particle is small compared to the typical length over which macroscopic quantities, such as the density, vary. In this problem, we check the validity of the fluid approximation by estimating the mean free path in some astrophysical situations.

The mean free path in elementary kinetic theory is given by:

$$\lambda = \frac{1}{n\sigma} \tag{1}$$

where n is the number of particles per unit volume and σ is the average cross section per particle. Use this formula to estimate the mean free path of

(a) The Warm Neutral Medium (WNM), one of the possible phases of the ISM, for which $n \sim 0.1\,\mathrm{cm}^{-3}$.

Hint: the main constituents of the WNM are atomic hydrogen and atomic helium. Since collisions between *neutral* atoms do not involve a strong long-range interaction, the collision cross-section is given approximately by $\sigma \simeq \pi(r_1^2 + r_2^2)$, where r_1 and r_2 are the radius of the two atoms involved in a collision.

(b) The Cold Neutral Medium (CNM), another possible phase of the ISM, for which $n \sim 100\,\mathrm{cm}^{-3}$.

Hint: the main constituent of the CNM is molecular hydrogen, H_2.

(c) A protostellar accretion disk, for which $n \sim 10^{10}\,\mathrm{cm}^{-3}$.

(d) Particles in the solar corona. In the solar corona, $n \sim 10^7\,\mathrm{cm}^{-3}$, $T \sim 10^6\,\mathrm{K}$.

Hint: the solar corona is almost completely ionised. Charged particles interact via the Coulomb force over distances much larger than atomic radii, which enhances the cross section as compared to hard sphere collisions. Thus the formula that we used in previous items to calculate cross sections for neutral particles is not valid, and you need to come up (or find in some book) an appropriate formula.

(e) Particles in the solar wind at a distance of 1 AU from the Sun (i.e., at around the location of the Earth). At this location, typical values are $n \sim 10\,\mathrm{cm}^{-3}$, $T \sim 10^5\,\mathrm{K}$.

Note: remember this result when we will discuss the Parker wind!

(f) Gas in a galaxy cluster. Plausible values for the density and temperature of gas within a galaxy cluster are $n \sim 10^{-3}\,\mathrm{cm}^{-3}$ and $T \sim 10^8\,\mathrm{K}$.

Hint: it follows from the high value of the temperature that the gas in a galaxy cluster is highly ionised.

2 Conservation of energy in an external gravitational potential.

We have seen that the following is a statement energy conservation for an adiabatic fluid subject only to pressure forces:

$$\partial_t \left[\frac{\rho \mathbf{v}^2}{2} + \frac{P}{\gamma - 1} \right] + \boldsymbol{\nabla} \cdot \left[\left(\frac{\rho \mathbf{v}^2}{2} + P + \frac{P}{\gamma - 1} \right) \mathbf{v} \right] = 0. \tag{2}$$

Show that the analogous statement in the presence of a given static external gravitational field $\Phi(\mathbf{x})$ is

$$\partial_t \left[\frac{\rho \mathbf{v}^2}{2} + \frac{P}{\gamma - 1} + \rho\Phi \right] + \boldsymbol{\nabla} \cdot \left[\left(\frac{\rho \mathbf{v}^2}{2} + P + \frac{P}{\gamma - 1} + \rho\Phi \right) \mathbf{v} \right] = 0. \tag{3}$$

Hint: start from the Euler equation

$$\partial_t \mathbf{v} + (\mathbf{v} \cdot \boldsymbol{\nabla})\mathbf{v} = -\frac{\boldsymbol{\nabla} P}{\rho} - \boldsymbol{\nabla}\Phi. \tag{4}$$

and follow the derivation in the lecture notes with the appropriate differences.

3 Lagrangian derivative of line and volume elements.

(a) Show that the Lagrangian derivative of a short line element is

$$\frac{D(\mathrm{d}\mathbf{l})}{Dt} = (\mathrm{d}\mathbf{l} \cdot \boldsymbol{\nabla})\mathbf{v} \tag{5}$$

which means that in a time $\mathrm{d}t$ it changes as $\mathrm{d}\mathbf{l} \to \mathrm{d}\mathbf{l} + (\mathrm{d}\mathbf{l} \cdot \boldsymbol{\nabla})\mathbf{v}\mathrm{d}t$.

Hint: consider how the endpoints move from t to $t + \mathrm{d}t$.

(b) Show that the Lagrangian derivative of a volume element is

$$\frac{D(\mathrm{d}V)}{Dt} = (\mathrm{d}V)\boldsymbol{\nabla} \cdot \mathbf{v} \tag{6}$$

which means that in a time $\mathrm{d}t$ the volume of a fluid element changes as $\mathrm{d}V \to \mathrm{d}V(1 + (\boldsymbol{\nabla} \cdot \mathbf{v})\mathrm{d}t)$, and $\boldsymbol{\nabla} \cdot \mathbf{v}$ is its rate of change.

Hint: consider a parallelepiped with edges $\mathrm{d}\mathbf{x} = \mathrm{d}x\hat{\mathbf{e}}_x$, $\mathrm{d}\mathbf{y} = \mathrm{d}y\hat{\mathbf{e}}_y$, $\mathrm{d}\mathbf{z} = \mathrm{d}z\hat{\mathbf{e}}_z$ and use the result of the previous point and that the volume of a parallelepiped is $\mathrm{d}V = |\mathrm{d}\mathbf{x} \cdot (\mathrm{d}\mathbf{y} \times \mathrm{d}\mathbf{z})|$.

Solutions

1 Validity of the fluid approximation

(a) The radius of a hydrogen atom is of the order of the Bohr radius, $a_0 = 5.29 \times 10^{-9}$ cm. Hence the cross section is approximately given by $\sigma \simeq \pi(2a_0)^2 \simeq 5 \times 10^{-16}\,\mathrm{cm}^2$. The mean free path in the WNM can then be estimated as

$$\lambda = \frac{1}{n\sigma} \simeq 2 \times 10^{16}\,\mathrm{cm} \simeq 1000\,\mathrm{AU} \tag{7}$$

which is much smaller than typical length-scales of interest, such as the scale-height of the gaseous Galactic disk ($\sim 100\,\mathrm{pc}$) or the size of typical atomic clouds ($\sim 10\,\mathrm{pc}$).

(b) The size of an H_2 molecule can be (very roughly) approximated as twice the size of an H atom, hence the cross section is approximately $\sigma \simeq 10^{-15}\,\mathrm{cm}^2$. This gives a mean free path of

$$\lambda = \frac{1}{n\sigma} \simeq 10^{13}\,\mathrm{cm} \simeq 1\,\mathrm{AU}. \tag{8}$$

(c) Using the same cross section estimated in the previous item, we find

$$\lambda = \frac{1}{n\sigma} \simeq 10^{5}\,\mathrm{cm} \simeq 10^{-8}\,\mathrm{AU}. \tag{9}$$

This is much smaller than the typical size of a protostellar disk ($\gtrsim 1\,\mathrm{AU}$).

(d) To determine the mean free path, we need to know the collision cross-section for ions, which interact via the Coulomb force. An accurate calculation of σ for e.g. electron-electron collisions can be found in standard textbooks, but can be estimated (very roughly) as follows. We define an effective interaction radius by equating the potential energy of one electron in the field of the other with the kinetic energy of the electron. Ignoring unimportant numerical factors, we have

$$\frac{e^2}{r_\mathrm{e}} \sim m_\mathrm{e} v_\mathrm{e}^2. \tag{10}$$

Since the kinetic energy of the electron is comparable to kT, we can also write this as

$$\frac{e^2}{r_\mathrm{e}} \sim kT, \tag{11}$$

and hence

$$\sigma \sim \pi r_\mathrm{e}^2 \sim \frac{e^4}{(kT)^2} \sim 10^{-5} T^{-2}\,\mathrm{cm}^2. \tag{12}$$

Using $T = 10^6$ K and $n = 10^7\,\mathrm{cm}^{-2}$ leads to a mean free path of

$$\lambda = \frac{1}{n\sigma} \simeq 10^{10}\,\mathrm{cm} \simeq 0.1\,\mathrm{R}_\odot. \tag{13}$$

A more precise calculation gives a larger value by a factor of $\log\Lambda \sim 20$, where $\log\Lambda$ is the Coulomb logarithm. The more precise calculation takes into account that i) most of the exchange of momentum happens cumulatively through many small distant interactions rather than few close strong encounters ii) electric forces are screened at large distances, and the interaction is cut off at the **Debye length**.

(e) Repeating the calculations of the previous item with $T = 10^5\,\mathrm{K}$ and $n = 10\,\mathrm{cm}^{-2}$ gives

$$\lambda = \frac{1}{n\sigma} \simeq 10^{14}\,\mathrm{cm} \simeq 5\,\mathrm{AU}. \tag{14}$$

Even with the correction factor of $\log\Lambda \sim 20$, this is quite large compared to the typical scale of interest ($\sim 1\,\mathrm{AU}$).

(f) Using the formula for the cross section (12) we find

$$\lambda = \frac{1}{n\sigma} \simeq 10^{24}\,\mathrm{cm} \simeq 300\,\mathrm{kpc}. \tag{15}$$

which is short with respect to the typical distances of interest in clusters ($\sim$ 1 Mpc), especially if we add the Coulomb logarithm correction factor.

2 Conservation of energy in an external gravitational potential.

Taking the dot product of $\mathbf{v}$ with the Euler equation, the LHS becomes

$$\mathbf{v}\cdot[\partial_t\mathbf{v} + (\mathbf{v}\cdot\boldsymbol{\nabla})\mathbf{v}] = \frac{1}{2}\left[\partial_t\mathbf{v}^2 + (\mathbf{v}\cdot\boldsymbol{\nabla})\mathbf{v}^2\right] \tag{16}$$

$$= \frac{1}{2}\frac{D\mathbf{v}^2}{Dt} \tag{17}$$

Putting this back together with the RHS we find

$$\frac{1}{2}\frac{D\mathbf{v}^2}{Dt} = \mathbf{v}\cdot\left[-\frac{\boldsymbol{\nabla}P}{\rho} - \boldsymbol{\nabla}\Phi\right] \tag{18}$$

After multiplying by ρ on both sides, we obtain:

$$\frac{1}{2}\rho\frac{D\mathbf{v}^2}{Dt} = \rho\mathbf{v}\cdot\left[-\frac{\boldsymbol{\nabla}P}{\rho} - \boldsymbol{\nabla}\Phi\right] \tag{19}$$

we can rewrite the LHS as follows

$$\frac{1}{2}\rho\frac{D\mathbf{v}^2}{Dt} = \frac{1}{2}\rho\left(\partial_t\mathbf{v}^2 + \mathbf{v}\cdot\boldsymbol{\nabla}\mathbf{v}^2\right) \tag{20}$$

$$= \frac{1}{2}\rho\left(\partial_t\mathbf{v}^2 + \mathbf{v}\cdot\boldsymbol{\nabla}\mathbf{v}^2\right) + \frac{1}{2}\mathbf{v}^2\left[\partial_t\rho + \boldsymbol{\nabla}\cdot(\rho\mathbf{v})\right] \tag{21}$$

$$= \partial_t\left(\frac{\rho\mathbf{v}^2}{2}\right) + \boldsymbol{\nabla}\left(\frac{\rho\mathbf{v}^2}{2}\mathbf{v}\right), \tag{22}$$

where in the second step we have used the continuity equation. Now let us take care of the RHS. The first term on the RHS of (19) can be rewritten as:

$$-(\mathbf{v}\cdot\boldsymbol{\nabla})\,P = -\boldsymbol{\nabla}\cdot(P\mathbf{v}) + P(\boldsymbol{\nabla}\cdot\mathbf{v}) \tag{23}$$

and the second term

$$-\rho\,(\mathbf{v}\cdot\boldsymbol{\nabla})\,\Phi = -\boldsymbol{\nabla}\cdot(\Phi\rho\mathbf{v}) + \Phi[\boldsymbol{\nabla}\cdot(\rho\mathbf{v})] \tag{24}$$

and using the continuity equation we have

$$\Phi[\boldsymbol{\nabla}\cdot(\rho\mathbf{v})] = -\Phi\partial_t\rho = -\partial_t(\rho\Phi). \tag{25}$$

Putting all together into (19) and rearranging we find

$$\partial_t\left[\frac{\rho\mathbf{v}^2}{2} + \rho\Phi\right] + \boldsymbol{\nabla}\cdot\left[\left(\frac{\rho\mathbf{v}^2}{2} + P + \rho\Phi\right)\mathbf{v}\right] = P(\boldsymbol{\nabla}\cdot\mathbf{v}) \tag{26}$$

From this point the derivation proceeds exactly as in the lecture notes, where it is shown that for an adiabatic gas the term on the RHS can be rewritten as

$$\boldsymbol{\nabla}\cdot\mathbf{v} = -\frac{1}{P(\gamma-1)}\left[\partial_t P + \boldsymbol{\nabla}\cdot(P\mathbf{v})\right] \tag{27}$$

Substituting this into (26) and rearranging we finally find:

$$\partial_t\left[\frac{\rho\mathbf{v}^2}{2} + \frac{P}{\gamma-1} + \rho\Phi\right] + \boldsymbol{\nabla}\cdot\left[\left(\frac{\rho\mathbf{v}^2}{2} + P + \frac{P}{\gamma-1} + \rho\Phi\right)\mathbf{v}\right] = 0. \tag{28}$$

3 Lagrangian derivative of line and volume elements.

(a) Consider how the end points A and B of a line element $\mathrm{d}\mathbf{l}$ move during a time $\mathrm{d}t$ (see figure). It is clear that

$$\begin{aligned}
D(\mathrm{d}\mathbf{l}) &= \mathrm{d}\mathbf{l}(t+\mathrm{d}t) - \mathrm{d}\mathbf{l}(t) && (29)\\
&= [B(t+\mathrm{d}t) - A(t+\mathrm{d}t)] - [B(t) - A(t)] && (30)\\
&= [B(t+\mathrm{d}t) - B(t)] - [A(t+\mathrm{d}t) - A(t)] && (31)\\
&= \mathbf{v}(B)\mathrm{d}t - \mathbf{v}(A)\mathrm{d}t && (32)\\
&= \mathbf{v}(A+\mathrm{d}\mathbf{l})\mathrm{d}t - \mathbf{v}(A)\mathrm{d}t && (33)\\
&\simeq (\mathrm{d}\mathbf{l}\cdot\boldsymbol{\nabla})\mathbf{v}\mathrm{d}t && (34)
\end{aligned}$$

which is the result given in the text.

(b) During a time $\mathrm{d}t$, each of the edges change as follows:

$$\begin{aligned}
\mathrm{d}\mathbf{x}' &= \mathrm{d}\mathbf{x} + (\mathrm{d}\mathbf{x}\cdot\boldsymbol{\nabla})\mathbf{v}\mathrm{d}t = \mathrm{d}x\hat{\mathbf{e}}_x + \mathrm{d}x\cdot(\partial_x\mathbf{v})\mathrm{d}t && (35)\\
\mathrm{d}\mathbf{y}' &= \mathrm{d}\mathbf{y} + (\mathrm{d}\mathbf{y}\cdot\boldsymbol{\nabla})\mathbf{v}\mathrm{d}t = \mathrm{d}y\hat{\mathbf{e}}_y + \mathrm{d}y\cdot(\partial_y\mathbf{v})\mathrm{d}t && (36)\\
\mathrm{d}\mathbf{z}' &= \mathrm{d}\mathbf{z} + (\mathrm{d}\mathbf{z}\cdot\boldsymbol{\nabla})\mathbf{v}\mathrm{d}t = \mathrm{d}z\hat{\mathbf{e}}_z + \mathrm{d}z\cdot(\partial_z\mathbf{v})\mathrm{d}t && (37)\\
& && (38)
\end{aligned}$$

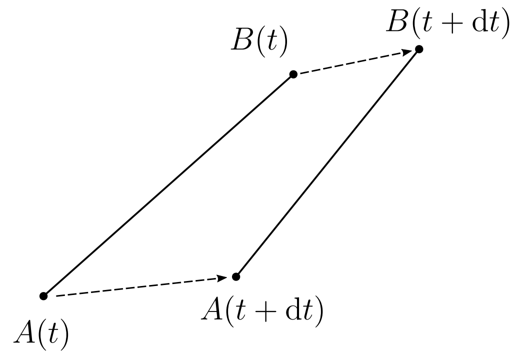

Figure 1: Evolution of a line element during an interval dt.

where $\mathbf{v}$ can be evaluated with negligible error at the center of the parallelepiped. By direct calculation we find that the new volume is

$$\begin{aligned}
\mathrm{d}V' &= |\mathrm{d}\mathbf{x}' \cdot (\mathrm{d}\mathbf{y}' \times \mathrm{d}\mathbf{z}')| && (39)\\
&\simeq |\mathrm{d}x\mathrm{d}y\mathrm{d}z \left\{1 + \mathrm{d}t \left[(\partial_x \mathbf{v}) \cdot (\hat{\mathbf{e}}_y \times \hat{\mathbf{e}}_z) + (\partial_y \mathbf{v}) \cdot (\hat{\mathbf{e}}_z \times \hat{\mathbf{e}}_x) + (\partial_z \mathbf{v}) \cdot (\hat{\mathbf{e}}_x \times \hat{\mathbf{e}}_y)\right]\right\}| && (40)\\
&= |\mathrm{d}x\mathrm{d}y\mathrm{d}z \left\{1 + \mathrm{d}t(\boldsymbol{\nabla} \cdot \mathbf{v})\right\}| && (41)\\
&= \mathrm{d}V(1 + \mathrm{d}t(\boldsymbol{\nabla} \cdot \mathbf{v})) && (42)
\end{aligned}$$

which is the wanted result.