Astrophysical Fluid Dynamics

Assignment #2: due May 16

1 Estimating Reynolds numbers

(a) Suppose we have a gas with temperature T and particle number density n that is composed of particles of mass m. This gas is flowing in the y direction with a bulk velocity

$$\mathbf{v} = v(x)\,\hat{\mathbf{e}}_y \tag{1}$$

that is a function of x only. Now consider a plane S located at $x = x_0$ (see figure). Since the gas is made of particles moving in random directions and colliding with each other, some particles will cross this plane, transporting momentum, even if the bulk velocity is completely in the y direction. We can assume that the particles that cross the plane are coming from within a layer whose thickness is roughly the mean free path λ.

Show that the amount of y component of momentum transported from left to right per unit area and per unit time, apart from numerical coefficients of order unity, can be estimated as

$$\dot{P}_y \simeq \frac{1}{6} n v_{\rm th} m \left[v(x_0) - \lambda \frac{\partial v}{\partial x}(x_0) \right], \tag{2}$$

where

$$v_{\rm th} = \sqrt{\frac{3kT}{m}} \tag{3}$$

is the thermal velocity of the gas particles. Show that the net flux, which can be obtained by also considering the corresponding expression for the momentum transport from right to left, is:

$$\Delta \dot{P}_y = \frac{1}{3} n v_{\rm th} m \lambda \frac{\partial v}{\partial x} \tag{4}$$

(b) Use your results from part (a) to show that we can estimate the coefficient of dynamic viscosity as

$$\eta = \frac{1}{3} n v_{\rm th} m \lambda. \tag{5}$$

which implies that

$$\eta \sim \frac{m v_{\rm th}}{\sigma}. \tag{6}$$

(c) Estimate the Reynolds number of the gas in:

(i) A protostellar accretion disk

(ii) A giant molecular cloud

(iii) Gas in a galaxy cluster

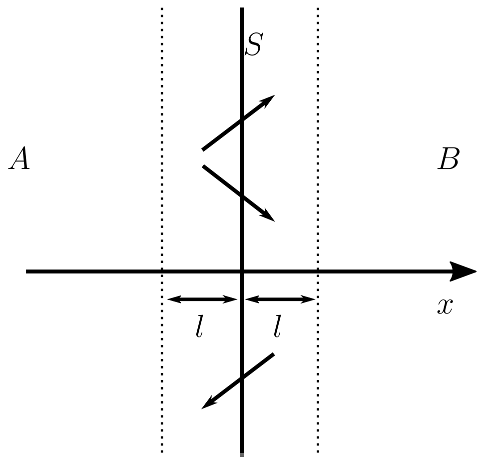

Figure 1: The microscopic origin of viscosity.

Hint: you can use the results of the previous exercise, but you will also need to search (e.g. in books, or the internet) for typical values of some quantities.

2 Rotating liquid

Consider a constant density, incompressible fluid rotating inside a container at a constant angular speed ω in a constant gravitational field g (see figure). The gravitational field is $\Phi = gz$. Find the shape of the liquid surface.

Can you think of any application of this result?

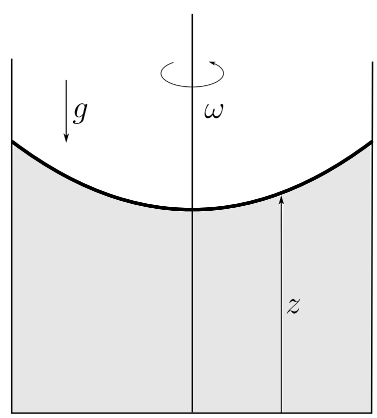

Figure 2: Rotating fluid in a container.

3 Bulging of the Earth

The radius of the Earth is slightly bigger at the equator than it is at the poles due to the centrifugal force arising from the rotation of the Earth around its axis. The goal of this problem is to find the shape of the Earth, first incorrectly, and then correctly. We assume that the Earth is made of an incompressible fluid of constant density.

1. A common *incorrect* method assumes that the gravitational potential of the Earth can be approximated by that of a sphere. One can then find the shape of the Earth by finding the equipotential surfaces in Earth's rotating frame (including the centrifugal potential). Show that this method leads to a surface whose height is given by

$$R = R_0 \left[1 - \left(\frac{R_0 \omega^2}{3g} \right) P_2(\cos\theta) \right] \tag{7}$$

   where

$$P_2(\cos\theta) = \frac{1}{2} \left(3\cos^2\theta - 1 \right) \tag{8}$$

   is the second Legendre polynomial, R_0 is the radius of the Earth if it were spherical, θ is the angle measured from the poles ($\theta = 90°$ is the equator), $g = GM/R_0^2$ is the gravitational potential at R_0 and ω is the angular velocity of the Earth.

2. The above method is incorrect, because the distortion of the Earth slightly changes its gravitational potential, which changes the equipotential surfaces. Turns out that this effect is of the same order of the one found in the previous item and cannot be neglected. Assuming that answer is of the form

$$R = R_0 \left[1 - \beta \left(\frac{R_0 \omega^2}{3g} \right) P_2(\cos\theta) \right] \tag{9}$$

   Calculate the correct value of β. What is the difference between the radius of the Earth at the equator and at the poles?
   Hint: the gravitational potential of a thin spherical shell of radius R_0 and surface density

$$\sigma(\theta) = \sigma_0 P_2(\cos\theta) \tag{10}$$

   where σ_0 is a constant, is given by (can you prove this?)

$$\Phi_2(r,\theta) = 4\pi G\,\sigma(\theta) \times \begin{cases} -\frac{1}{5} R_0^{-1} r^2 & \text{if } r \le R_0 \\ -\frac{1}{5} R_0^4 r^{-3} & \text{if } r \ge R_0 \end{cases}$$

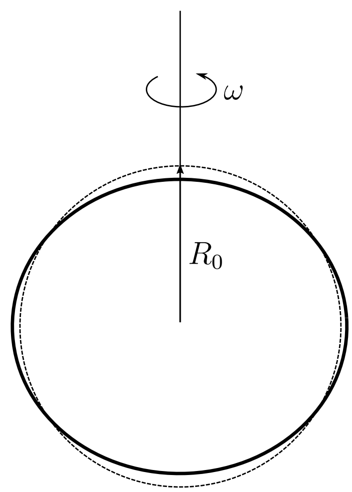

Figure 3: Schematic illustration of the bulging of the Earth. The dashed circle is the shape of the Earth if it were not spinning. The solid line represents the actual shape.

Solutions

1 Estimating Reynolds numbers

(a) We can assume that roughly one third of the particles are moving in the x direction, and half of these will be moving from in the positive x direction and the other half in the negative x direction. We can also assume that the mean speed of these particles is given by $v_{\rm th}$. The flux of particles across the plane $x = x_0$ will therefore be

$$\text{Flux} \simeq \frac{1}{6} n v_{\rm th}. \tag{11}$$

The average value of the y component of momentum of each of these particles at a position $x_0 - \lambda$ is equal to

$$\bar{p}_y = m\left[v(x_0 - \lambda)\right] \simeq m\left[v(x_0) - \lambda \frac{\partial v}{\partial x}(x_0)\right], \tag{12}$$

where we have assumed that the characteristic length scale over which v changes is much greater than λ (which is required if the fluid approximation is valid). The flux of the x component of momentum from left to right is therefore

$$\dot{P}_y = \frac{1}{6} n v_{\rm th} \bar{p}_x = \frac{1}{6} n v_{\rm th} m \left[v(x_0) - \lambda \frac{\partial v}{\partial y}(x_0)\right]. \tag{13}$$

Similarly, for particles moving from right to left we have

$$\dot{P}_y = \frac{1}{6} n v_{\rm th} \bar{p}_{x,-} = \frac{1}{6} n v_{\rm th} m \left[v_{\rm x,0} + \lambda \frac{\partial v}{\partial x}(x_0)\right]. \tag{14}$$

The net flux of momentum across the boundary is then

$$\Delta \dot{P}_y = \frac{1}{3} n v_{\rm th} m \lambda \frac{\partial v}{\partial x} \tag{15}$$

(b) The flux of momentum is the same thing as the rate of change of momentum per unit area, or in other words the force per unit area. We therefore have:

$$\frac{F_x}{A} = \Delta \dot{P}_y. \tag{16}$$

Finally, the shear viscosity is defined by the expression

$$\frac{F_x}{A} = \eta \frac{\partial v_x}{\partial y}, \tag{17}$$

and so it follows that

$$\eta = \frac{1}{3} n v_{\rm th} m \lambda. \tag{18}$$

Using that $\lambda = (n\sigma)^{-1}$ we have

$$\eta \sim \frac{1}{3} \frac{v_{\rm th} m}{\sigma} \sim \frac{v_{\rm th} m}{\sigma} \tag{19}$$

(c) (i) For the outer part of the disk, plausible numbers are $L = 100$ AU, $v = 3\,\mathrm{km\,s^{-1}}$, $T = 30$ K and $n = 10^{10}\,\mathrm{cm^{-3}}$. Using these values, we get

$$v_{\rm th} = \sqrt{\frac{3kT}{m_{\rm H}}} \sim 10^5\ \mathrm{cm\,s^{-1}}, \tag{20}$$

$$\eta = \frac{v_{\rm th} m_{\rm H}}{\sigma} \sim 1.5\times 10^{-4}\ \mathrm{g\,cm^{-1}\,s^{-1}}, \tag{21}$$

$$\nu = \frac{\eta}{\rho} \sim 10^{10}\ \mathrm{cm^2\,s^{-1}}, \tag{22}$$

and so

$$\mathrm{Re} = \frac{vL}{\nu} \sim 4.5\times 10^{10}. \tag{23}$$

which is fairly high, indicating that the disk will be turbulent.

(ii) From Larson's first "law", we know that $v \sim 1\,(L/1\,\mathrm{pc})^{1/2}\,\mathrm{km\,s^{-1}}$ within a typical molecular cloud. If we take $L = 10\,\mathrm{pc}$ as a sensible order of magnitude estimate for the size of a GMC, we have $v \sim 3\,\mathrm{km\,s^{-1}} = 3\times 10^5\ \mathrm{cm\,s^{-1}}$. The coefficient of shear viscosity is approximately $\eta \sim m v_{\rm t} \sigma^{-1}$, the thermal velocity within the cold molecular gas is $\sim 0.2\,\mathrm{km\,s^{-1}}$, and $\sigma \sim 10^{-15}\ \mathrm{cm^2}$. Therefore,

$$\eta = \frac{v_{\rm th} m_{\rm H}}{\sigma} \sim 3.5\times 10^{-5}\ \mathrm{g\,cm^{-1}\,s^{-1}}. \tag{24}$$

If we adopt $n \sim 1000\,\mathrm{cm^{-3}}$ as a typical number density for the gas inside a GMC, then we have

$$\nu = \eta/\rho \simeq 2\times 10^{16}\ \mathrm{cm^2\,s^{-1}}, \tag{25}$$

from which it follows that

$$\mathrm{Re} = \frac{vL}{\nu} \sim 5\times 10^8. \tag{26}$$

(iii) Using $n \sim 10^{-3}\,\mathrm{cm^{-3}}$ and $T \sim 10^8\,\mathrm{K}$ and the same expression for the cross-section derived in the solution of exercise **??**

$$\sigma \sim \pi r_{\rm e}^2 \sim \frac{e^4}{(kT)^2} \sim 10^{-5} T^{-2}\ \mathrm{cm^2}. \tag{27}$$

For $T = 10^8\,\mathrm{K}$ we find $\sigma = 10^{-21}$ cm. A more precise calculation gives a larger value by a factor of $\log\Lambda \sim 40$, where $\log\Lambda$ is the Coulomb logarithm. The thermal velocity of the electrons at this temperature is $v_{\rm th} \sim 4\times 10^9\ \mathrm{cm\,s^{-1}}$, and the electrons therefore have a dynamic viscosity

$$\eta = \frac{v_{\rm th} m_{\rm e}}{\sigma} \sim 100\ \mathrm{g\,cm^{-1}\,s^{-1}}. \tag{28}$$

A similar analysis for the ions gives a value that is larger by a factor of $(m_{\rm H}/m_{\rm e})^{1/2} \sim 40$, so for the plasma as a whole we have $\eta \sim 4000$. The kinematic viscosity is then

$$\nu = \frac{\eta}{\rho} \sim 10^{30}\ \mathrm{cm^2\,s^{-1}}. \tag{29}$$

Finally, to compute the Reynolds number, we need characteristic numbers for v and L. If we take $L = 1\,\mathrm{Mpc}$ and $v = 1000\,\mathrm{km\,s^{-1}}$ as reasonable values, then

$$\mathrm{Re} = \frac{vL}{\nu} = \frac{10^8 \times (3 \times 10^{24})}{10^{30}} \sim 300. \tag{30}$$

The Reynolds number for the hot plasma in a galaxy cluster is therefore fairly small.

2 Rotating liquid

After transients have been dissipated away, we can assume that the fluid will be stationary in the rotating frame of the vessel. The shape of the fluid will then follow equipotential surfaces of the total effective potential, which is the sum of the gravitational potential plus the centrifugal potential

$$\Phi_{\mathrm{eff}} = gz - \frac{1}{2}\omega R^2 \tag{31}$$

where R is the distance from the axis of rotation in polar coordinates. If the surface of a fluid at rest in the rotating frame did not follow the equipotential surfaces, then fluid elements on the surface would feel a net force, putting it in motion. Solving for z, we see that the surfaces $\Phi_{\mathrm{eff}} = \mathrm{constant}$ are of the form

$$\boxed{z = z_0 + \frac{1}{2}\omega R^2} \tag{32}$$

which means that the fluid surface is a parabola. Alternatively, one can obtain the same result by using the Euler equation in a rotating frame. Assuming steady state, one obtains:

$$0 = -\frac{\boldsymbol{\nabla} P}{\rho} + \omega^2 R\hat{\mathbf{e}}_R - g\hat{\mathbf{e}}_z \tag{33}$$

Solving for P gives:

$$P = \frac{1}{2}\rho\omega^2 R^2 - gz + \mathrm{constant} \tag{34}$$

The liquid surface is a surface of constant pressure. Imposing this, we obtain the same result as before.

There is an interesting application of this result. Parabolic mirrors have the characteristic that they reflect all parallel rays coming from infinity into a single point, the focus. This is exactly the property one is looking for when building telescopes! Thus, we can build telescope mirrors by spinning liquids and make them solidify while spinning, for example putting them in big ovens. Or, we can even have liquid mirror telescopes, which are telescopes with mirrors made with a reflective liquid such as mercury. Very large telescopes can also be constructed (and controlled) by using many small mirrors, and these mirrors segments can be constructed by spinning small portions of fluid.

3 Bulging of the Earth

1. Equipotential surfaces in the rotating frame of the Earth are defined by

$$\frac{GM}{r} - \frac{1}{2}\omega^2 r^2 \sin^2\theta = A \tag{35}$$

where A is a constant to be determined. Substituting $r = R_0 + R_1$ and keeping only first order terms in the small quantities R_1/R_0 and in $R_0\omega^2/g$ we find

$$R_1 \simeq B + \frac{1}{2}\frac{\omega^2 R_0^2}{g}\sin^2\theta \tag{36}$$

where B is another constant which includes A and all the constant terms coming from the expansion. We can determine B by requiring that the total volume of the Earth is unchanged. The extra volume due to R_1 is approximately given by

$$\begin{aligned}
\Delta V &= R_0^2 \int R_1 \mathrm{d}\Omega & (37)\\
&= R_0^2 \int R_1 \mathrm{d}\sin\theta \mathrm{d}\theta \mathrm{d}\phi & (38)\\
&= 2\pi R_0^2 \int_0^\pi \left(B + \frac{1}{2}\frac{\omega^2 R_0^2}{g}\sin^2\theta\right)\sin\theta \mathrm{d}\theta & (39)\\
&= 4\pi R_0^2 \left(B + \frac{\omega^2 R_0^2}{3g}\right) & (40)
\end{aligned}$$

where integral is extended over the whole solid angle. Requiring $\Delta V = 0$ we find

$$B = -\frac{\omega^2 R_0^2}{3g} \tag{41}$$

Using this result and $\sin^2\theta = 1 - \cos^2\theta$ we can find the result given in the text

$$\boxed{R_1 = -\frac{\omega^2 R_0^2}{3g}\left[\frac{1}{2}\left(3\cos^2\theta - 1\right)\right]} \tag{42}$$

2. Let us define

$$R_1 = R - R_0 = -\beta R_0 \left(\frac{R_0\omega^2}{3g}\right) P_2(\cos\theta) \tag{43}$$

We model the Earth as the sum of a sphere of constant density plus a thin shell with positive or negative mass, depending on the sign of R_1 at a given location. The surface density of this shell is

$$\begin{aligned}
\sigma(\theta) &= \rho R_1 & (44)\\
&= -\beta\rho R_0 \left(\frac{R_0\omega^2}{3g}\right) P_2(\cos\theta) & (45)
\end{aligned}$$

where ρ is the density of the Earth, which we assume constant, hence

$$\rho = \frac{M}{V} = \frac{M}{(4/3)\pi R_0^3} \tag{46}$$

and we can rewrite the surface density as

$$\sigma(\theta) = -\frac{3M}{4\pi R_0^2}\beta\left(\frac{R_0\omega^2}{3g}\right)P_2(\cos\theta) \tag{47}$$

This shell gives rise to a quadrupole potential whose functional form is given in the hint in the text. For $r \geq R_0$ it is:

$$\begin{aligned}\Phi_2(r,\theta) &= -\frac{4\pi G}{5}\sigma R_0^4 r^{-3} &(48)\\ &= \frac{3}{5}\beta g\left(\frac{R_0\omega^2}{3g}\right)R_0^4 r^{-3}P_2(\cos\theta) &(49)\end{aligned}$$

This already contains the small parameter $(R_0\omega^2/3g)$, hence at the Earth surface we can approximate it by putting $r \simeq R_0$, hence

$$\Phi_2(r,\theta) \simeq \frac{3}{5}\beta g R_0\left(\frac{R_0\omega^2}{3g}\right)P_2(\cos\theta) \tag{50}$$

To find the shape of the Earth we need to find the equipotential surfaces of the full potential that includes Φ_2:

$$\frac{GM}{r} - \frac{1}{2}\omega^2 r^2\sin^2\theta + \Phi_2(r,\theta) = A \tag{51}$$

where A is a constant to be determined. Using $R = R_0 + R_1$ and expanding and neglecting small terms, we can rewrite this as

$$gR_1 + \frac{1}{3}\omega^2 R_0^2 P_2(\cos\theta) + \Phi_2(r,\theta) = B \tag{52}$$

where B is another constant. Note that we have manipulated the centrifugal term to make $P_2(\cos\theta)$ appear by adding and subtracting a constant, which then ends up inside B. Now substituting (50) and (43) we obtain

$$\left[-\beta g R_0\left(\frac{R_0\omega^2}{3g}\right) + \frac{1}{3}\omega^2 R_0^2 + \frac{3}{5}\beta g R_0\left(\frac{R_0\omega^2}{3g}\right)\right]P_2(\cos\theta) = B \tag{53}$$

The left hand side depends on θ, while the right hand side doesn't. The only way this equation can be satisfied is if the left hand side vanishes. This is also consistent with requiring that the volume of the Earth remains unchanged, which requires $B = 0$. We find

$$\boxed{\beta = \frac{5}{2}} \tag{54}$$

The difference between the Earth radius at the equation and at the poles is

$$h = \frac{3}{2}\beta\left(\frac{R_0\omega^2}{3g}\right) \simeq 27\,\mathrm{km} \tag{55}$$

The true value is $h = 21.5\,\mathrm{km}$. The fact that this is smaller than our value makes sense for the following reason. In reality, the Earth's density is not constant as we assumed, but decreases with radius and it is higher deeper inside the Earth and smaller at the surface. In the extreme case that all the Earth's mass were concentrated at the centre, then the thin distortion shell at the surface would have no effect on the potential, because it would be massless. So the naive calculation in the first part of the problem would in fact be correct, and we would obtain $h = 11\,\mathrm{km}$. In the real case the density lies somewhere between the case of a concentrated centre and the case of uniform density. The actual value of h should therefore lie somewhere between the corresponding h values of 11 km and 28 km. And 21.5 km indeed does.

This calculation was performed for the first time by Newton in the first edition of the *Principia*, where he obtained the value we found in the second part of the calculations! At the time, this lead to a dispute between Newton and Cassini, because the latter believed that the Earth radius was greater at the poles, the opposite of the actual result.