Astrophysical Fluid Dynamics

Assignment #3: due November 6

1 Galilean invariance of the MHD equations.

In this problem we want to prove that the ideal MHD equations:

$$\partial_t \rho + \boldsymbol{\nabla} \cdot (\rho \mathbf{v}) = 0 \tag{1}$$

$$\rho \frac{D\mathbf{v}}{Dt} = -\boldsymbol{\nabla}\left(P + \frac{B^2}{8\pi}\right) - \boldsymbol{\nabla}\Phi + \frac{1}{4\pi}\left(\mathbf{B}\cdot\boldsymbol{\nabla}\right)\mathbf{B} \tag{2}$$

$$\frac{D}{Dt}\left(\log P\rho^{-\gamma}\right) = 0. \tag{3}$$

$$\partial_t \mathbf{B} = \boldsymbol{\nabla} \times (\mathbf{v} \times \mathbf{B}) \tag{4}$$

are Galilean Invariant, i.e. they are invariant under a transformation of the coordinates of the following type:

$$\mathbf{x}' = \mathbf{x} - \mathbf{v}_0 t, \quad t' = t. \tag{5}$$

1. Show that the appropriate transformation laws for the electric and magnetic fields under the ideal MHD approximations are:

$$\mathbf{E}' = \mathbf{E} + \frac{\mathbf{v}_0}{c} \times \mathbf{B} \tag{6}$$

$$\mathbf{B}' = \mathbf{B}. \tag{7}$$

   Hint: this can be done either by approximating the full relativistic formulas for the transformation of the fields, or by considering the Lorentz force on individual charged particles, $\mathbf{F} = q\mathbf{E} + (q/c)\mathbf{v}\times\mathbf{B}$. Requiring that the latter formula is valid as a function of the primed fields in the primed frame (in which the particle velocity is $\mathbf{v}' = \mathbf{v} - \mathbf{v}_0$) for all possible values of $\mathbf{v}$, one obtains the same result.

2. Show that the appropriate law of transformation for $\mathbf{J}$ is:

$$\mathbf{J}' = \mathbf{J} \tag{8}$$

3. Show that gradient and the time derivative change according to

$$\boldsymbol{\nabla}' = \boldsymbol{\nabla}, \qquad \partial_{t'} = \partial_t + \mathbf{v}_0 \cdot \boldsymbol{\nabla} \tag{9}$$

4. Finally, use the results of the preceding steps to prove that the ideal MHD equations (1), (2), (3) and (4) are Galilean Invariant.

2 The limit of ideal MHD in astrophysical situations

We have seen in the lectures that the induction equation is

$$\partial_t \mathbf{B} = \boldsymbol{\nabla} \times (\mathbf{v} \times \mathbf{B}) - \boldsymbol{\nabla} \times \left(\frac{c^2}{4\pi\sigma} \boldsymbol{\nabla} \times \mathbf{B} \right). \tag{10}$$

When the second term on the RHS of this equation can be neglected, we are in the limit of ideal MHD. The importance of the second term can be estimated noting that if we neglect the first term on the RHS and consider σ a constant, the induction equation can be rewritten as

$$\partial_t \mathbf{B} = \frac{c^2}{4\pi\sigma} \boldsymbol{\nabla}^2 \mathbf{B}. \tag{11}$$

This is the familiar **diffusion equation**, also known as heat equation. It is well known that this leads to a diffusion timescale of

$$t_d = \frac{4\pi\sigma L^2}{c^2}, \tag{12}$$

where L is a characteristic size of the system. The greater t_d, the closer we are to the limit of ideal MHD. The coefficient $c^2/(4\pi\sigma)$ is called magnetic diffusivity.

The conductivity of a completely ionised gas of pure hydrogen can be written as

$$\sigma = 6.98 \times 10^7 \frac{T^{3/2}}{\log \Lambda} \, \mathrm{s}^{-1} \tag{13}$$

Using the formulas above, estimate the diffusion time scale in

(a) The interior of the Sun

(b) A Galactic HII region

(c) The intergalactic medium at redshift $z = 0$

In which of these systems is flux freezing a good approximation?

3 Amplification of magnetic fields

In this problem we want to see how the magnetic field can be amplified in ideal MHD by considering its evolution in a *prescribed* velocity field $\mathbf{v}$.

1. Consider the evolution of an initially uniform field $\mathbf{B}_0 = B_0 \hat{\mathbf{e}}_x$ in the following stationary converging-diverging flow:

$$v_x = \alpha x, \quad v_y = -\alpha y, \quad v_z = 0, \tag{14}$$

where α is a constant. Assume that the fluid density ρ is uniform at $t = 0$. Show that the field remains uniform and grows as

$$\mathbf{B}(t) = \mathbf{B}_0 \exp(\alpha t) \tag{15}$$

and sketch the velocity and magnetic field lines.

2. Find the time evolution of an initially uniform field $\mathbf{B}_0 = B_0 \hat{\mathbf{e}}_x$ in the following shearing velocity field:

$$v_x = 0, \quad v_y = \begin{cases} -v_0 & \text{if } x < -a \\ v_0(x/a) & \text{if } -a \leq x \leq a \\ v_0 & \text{if } x > a \end{cases}, \quad v_z = 0. \tag{16}$$

where v_0 and a are constants. Assume that the fluid density ρ is uniform at $t = 0$. Sketch the velocity and magnetic field lines.

3. In the preceding examples the energy stored into magnetic fields increases with time. Where is this energy coming from?