Astrophysical Fluid Dynamics

Assignment #3: due May 23rd

1 Galilean invariance of the MHD equations.

In this problem we want to prove that the ideal MHD equations:

$$\partial_t \rho + \boldsymbol{\nabla} \cdot (\rho \mathbf{v}) = 0 \tag{1}$$

$$\rho \frac{D\mathbf{v}}{Dt} = -\boldsymbol{\nabla} \left(P + \frac{B^2}{8\pi} \right) - \boldsymbol{\nabla}\Phi + \frac{1}{4\pi} \left(\mathbf{B} \cdot \boldsymbol{\nabla} \right) \mathbf{B} \tag{2}$$

$$\frac{D}{Dt} \left(\log P \rho^{-\gamma} \right) = 0. \tag{3}$$

$$\partial_t \mathbf{B} = \boldsymbol{\nabla} \times (\mathbf{v} \times \mathbf{B}) \tag{4}$$

are Galilean Invariant, i.e. they are invariant under a transformation of the coordinates of the following type:

$$\mathbf{x}' = \mathbf{x} - \mathbf{v}_0 t, \quad t' = t. \tag{5}$$

1. Show that the appropriate transformation laws for the electric and magnetic fields under the ideal MHD approximations are:

$$\mathbf{E}' = \mathbf{E} + \frac{\mathbf{v}_0}{c} \times \mathbf{B} \tag{6}$$

$$\mathbf{B}' = \mathbf{B}. \tag{7}$$

   Hint: this can be done either by approximating the full relativistic formulas for the transformation of the fields, or by considering the Lorentz force on individual charged particles, $\mathbf{F} = q\mathbf{E} + (q/c)\mathbf{v} \times \mathbf{B}$. Requiring that the latter formula is valid as a function of the primed fields in the primed frame (in which the particle velocity is $\mathbf{v}' = \mathbf{v} - \mathbf{v}_0$) for all possible values of $\mathbf{v}$, one obtains the same result.

2. Show that the appropriate law of transformation for $\mathbf{J}$ is:

$$\mathbf{J}' = \mathbf{J} \tag{8}$$

3. Show that the gradient and the time derivative change according to

$$\boldsymbol{\nabla}' = \boldsymbol{\nabla}, \qquad \partial_{t'} = \partial_t + \mathbf{v}_0 \cdot \boldsymbol{\nabla} \tag{9}$$

4. Finally, use the results of the preceding steps to prove that the ideal MHD equations (1), (2), (3) and (4) are Galilean Invariant.

2 The limit of ideal MHD in astrophysical situations

We have seen in the lectures that the induction equation is

$$\partial_t \mathbf{B} = \boldsymbol{\nabla} \times (\mathbf{v} \times \mathbf{B}) - \boldsymbol{\nabla} \times \left(\frac{c^2}{4\pi\sigma} \boldsymbol{\nabla} \times \mathbf{B} \right). \tag{10}$$

When the second term on the RHS of this equation can be neglected, we are in the limit of ideal MHD. The importance of the second term can be estimated noting that if we neglect the first term on the RHS and consider σ a constant, the induction equation can be rewritten as

$$\partial_t \mathbf{B} = \frac{c^2}{4\pi\sigma} \boldsymbol{\nabla}^2 \mathbf{B}. \tag{11}$$

This is the familiar **diffusion equation**, also known as heat equation. It is well known that this leads to a diffusion timescale of

$$t_d = \frac{4\pi\sigma L^2}{c^2}, \tag{12}$$

where L is a characteristic size of the system. The greater t_d, the closer we are to the limit of ideal MHD. The coefficient $c^2/(4\pi\sigma)$ is called magnetic diffusivity.

The conductivity of a completely ionised gas of pure hydrogen can be written as

$$\sigma = 6.98 \times 10^7 \frac{T^{3/2}}{\log \Lambda} \, \mathrm{s}^{-1} \tag{13}$$

Where $\log \Lambda \simeq 30$ is the Coulomb logarithm and T is in units of Kelvin. Using the formulas above, estimate the diffusion time scale in

(a) The interior of the Sun

(b) A Galactic HII region

(c) The intergalactic medium at redshift $z = 0$

In which of these systems is flux freezing a good approximation?

3 Amplification of magnetic fields

In this problem we want to see how the magnetic field can be amplified in ideal MHD by considering its evolution in a *prescribed* velocity field $\mathbf{v}(\mathbf{x}, t)$.

1. Consider the evolution of an initially uniform field $\mathbf{B}_0 = B_0 \hat{\mathbf{e}}_x$ in the following stationary converging-diverging flow:

$$v_x = \alpha x, \quad v_y = -\alpha y, \quad v_z = 0, \tag{14}$$

where α is a constant. Assume that the fluid density ρ is uniform at $t = 0$. Show that the field remains uniform and grows as

$$\mathbf{B}(t) = \mathbf{B}_0 \exp(\alpha t) \tag{15}$$

and sketch the velocity and magnetic field lines.

2. Find the time evolution of an initially uniform field $\mathbf{B}_0 = B_0 \hat{\mathbf{e}}_x$ in the following shearing velocity field:

$$v_x = 0, \quad v_y = \begin{cases} -v_0 & \text{if } x < -a \\ v_0 (x/a) & \text{if } -a \le x \le a \\ v_0 & \text{if } x > a \end{cases}, \quad v_z = 0. \tag{16}$$

where v_0 and a are positive constants. Assume that the fluid density ρ is uniform at $t = 0$. Sketch the velocity and magnetic field lines.

3. In the preceding examples the energy stored into magnetic fields increases with time. Where is this energy coming from?

Solutions

1 Galilean invariance of the MHD equations.

1. We will follow the second path suggested in the text. The Lorentz force in the primed Frame reads:

$$\mathbf{F}' = q\mathbf{E}' + q\frac{\mathbf{v}'}{c} \times \mathbf{B}' \tag{17}$$
$$= q\mathbf{E}' + q\frac{\mathbf{v} - \mathbf{v}_0}{c} \times \mathbf{B}' \tag{18}$$
$$= q\left(\mathbf{E}' - q\frac{\mathbf{v}_0}{c} \times \mathbf{B}'\right) + q\frac{\mathbf{v}}{c} \times \mathbf{B}' \tag{19}$$

   Since in Newtonian mechanics forces are the same in all inertial frames,

$$\mathbf{F}' = \mathbf{F} \tag{20}$$

   Hence

$$q\left(\mathbf{E}' - \frac{\mathbf{v}_0}{c} \times \mathbf{B}'\right) + q\frac{\mathbf{v}}{c} \times \mathbf{B}' = q\mathbf{E} + q\frac{\mathbf{v}}{c} \times \mathbf{B} \tag{21}$$

   This must be valid for all possible $\mathbf{v}$. The only possibility is that

$$\mathbf{E}' = \mathbf{E} + \frac{\mathbf{v}_0}{c} \times \mathbf{B} \tag{22}$$
$$\mathbf{B}' = \mathbf{B} \tag{23}$$

2. The full relativistic transformation law for the current is

$$\mathbf{J}' = \mathbf{J} - \gamma\rho\mathbf{v} + (\gamma - 1)(\mathbf{J} \cdot \hat{\mathbf{v}})\,\hat{\mathbf{v}}\,, \tag{24}$$

   where $\gamma = 1/\sqrt{1 - v^2/c^2}$. We neglect terms of the order v^2/c^2. Therefore, we can put $\gamma \simeq 1$, and $\rho\mathbf{v} \simeq 0$ in the above formula. We get

$$\mathbf{J}' = \mathbf{J}\,. \tag{25}$$

   Alternatively, one could prove this starting from the fourth Maxwell equation, approximated as explained in the lecture notes ($\nabla \times \mathbf{B} = [4\pi/c]\mathbf{J}$).

3. We have

$$\frac{\partial}{\partial x'} = \frac{\partial x}{\partial x'}\frac{\partial}{\partial x} + \frac{\partial y}{\partial x'}\frac{\partial}{\partial y} + \frac{\partial z}{\partial x'}\frac{\partial}{\partial z} + \frac{\partial t}{\partial x'}\frac{\partial}{\partial t} \tag{26}$$
$$\frac{\partial}{\partial y'} = \frac{\partial x}{\partial y'}\frac{\partial}{\partial x} + \frac{\partial y}{\partial y'}\frac{\partial}{\partial y} + \frac{\partial z}{\partial x'}\frac{\partial}{\partial z} + \frac{\partial t}{\partial y'}\frac{\partial}{\partial t} \tag{27}$$
$$\frac{\partial}{\partial z'} = \frac{\partial x}{\partial z'}\frac{\partial}{\partial x} + \frac{\partial y}{\partial z'}\frac{\partial}{\partial y} + \frac{\partial z}{\partial x'}\frac{\partial}{\partial z} + \frac{\partial t}{\partial z'}\frac{\partial}{\partial t} \tag{28}$$
$$\frac{\partial}{\partial t'} = \frac{\partial x}{\partial t'}\frac{\partial}{\partial x} + \frac{\partial y}{\partial t'}\frac{\partial}{\partial y} + \frac{\partial z}{\partial x'}\frac{\partial}{\partial z} + \frac{\partial t}{\partial t'}\frac{\partial}{\partial t} \tag{29}$$

The only non-vanishing derivatives are:

$$\frac{\partial x}{\partial x'} = 1; \qquad \frac{\partial y}{\partial y'} = 1 \tag{30}$$

$$\frac{\partial z}{\partial z'} = 1; \qquad \frac{\partial t}{\partial t'} = 1 \tag{31}$$

$$\frac{\partial x}{\partial t'} = v_{0x}; \qquad \frac{\partial y}{\partial t'} = v_{0y}; \qquad \frac{\partial z}{\partial t'} = v_{0z} \tag{32}$$

$$\tag{33}$$

This yields

$$\frac{\partial}{\partial x'} = \frac{\partial}{\partial x} \tag{34}$$

$$\frac{\partial}{\partial y'} = \frac{\partial}{\partial y} \tag{35}$$

$$\frac{\partial}{\partial z'} = \frac{\partial}{\partial z} \tag{36}$$

$$\frac{\partial}{\partial t'} = v_{0x}\frac{\partial}{\partial x} + v_{0y}\frac{\partial}{\partial y} + v_{0z}\frac{\partial}{\partial z} + \frac{\partial}{\partial t} \tag{37}$$

which is the result stated in the text.

4. First let us prove that the convective derivative is Galilean invariant:

$$\partial_t + \mathbf{v}\cdot\boldsymbol{\nabla} = \partial_{t'} - \mathbf{v}_0\cdot\boldsymbol{\nabla} + \mathbf{v}\cdot\boldsymbol{\nabla} \tag{38}$$

$$= \partial_{t'} - \mathbf{v}_0\cdot\boldsymbol{\nabla}' + \mathbf{v}\cdot\boldsymbol{\nabla}' \tag{39}$$

$$= \partial_{t'} - \mathbf{v}_0\cdot\boldsymbol{\nabla}' + (\mathbf{v}' + \mathbf{v}_0)\cdot\boldsymbol{\nabla}' \tag{40}$$

$$= \partial_{t'} + \mathbf{v}'\cdot\boldsymbol{\nabla}' \tag{41}$$

Since $P' = P$ and $\rho' = \rho$, this automatically proves that (3) is Galilean invariant. The continuity equation can be rewritten as

$$\frac{D}{Dt}\rho + \rho\boldsymbol{\nabla}\cdot\mathbf{v} = 0 \tag{42}$$

The first term is Galilean invariant because the convective derivative and ρ are both invariant. For the second term we have $\rho\boldsymbol{\nabla}\cdot\mathbf{v} = \rho\boldsymbol{\nabla}\cdot(\mathbf{v}' + \mathbf{v}_0) = \rho'\boldsymbol{\nabla}'\cdot\mathbf{v}'$, so it is also invariant. Hence the continuity equation is invariant. Similarly, the LHS of the Euler equation is invariant (the derivatives of $\mathbf{v}_0$ are zero). The RHS of the Euler equation is also invariant, because every single term is invariant (B, $\boldsymbol{\nabla}$, P, Φ are all invariant). Finally, for the induction equation we have on the LHS

$$\partial_t\mathbf{B} = \partial_{t'} - \mathbf{v}_0\cdot\boldsymbol{\nabla}\mathbf{B} \tag{43}$$

and for the RHS

$$\boldsymbol{\nabla}\times(\mathbf{v}\times\mathbf{B}) = \boldsymbol{\nabla}\times((\mathbf{v}' + \mathbf{v}_0)\times\mathbf{B}) \tag{44}$$

$$= \boldsymbol{\nabla}\times(\mathbf{v}'\times\mathbf{B}) + \boldsymbol{\nabla}\times(\mathbf{v}_0\times\mathbf{B}) \tag{45}$$

Now using the identity

$$\varepsilon_{ijk}\varepsilon^{imn} = \delta_j{}^m\delta_k{}^n - \delta_j{}^n\delta_k{}^m \tag{46}$$

One can show that

$$\nabla \times (\mathbf{v}_0 \times \mathbf{B}) = \mathbf{v}_0(\nabla \cdot \mathbf{B}) - (\mathbf{v}_0 \cdot \nabla)\mathbf{B} \tag{47}$$

$$= -(\mathbf{v}_0 \cdot \nabla)\mathbf{B} \tag{48}$$

where in the second step we have used that the magnetic field is divergence-free. Putting all together we have that the RHS of the induction equation can be rewritten as

$$\nabla \times (\mathbf{v} \times \mathbf{B}) = \nabla' \times (\mathbf{v}' \times \mathbf{B}') - (\mathbf{v}_0 \cdot \nabla)\mathbf{B} \tag{49}$$

Equating this with (43) we see that the term $(\mathbf{v}_0 \cdot \nabla)\mathbf{B}$ simplifies, which concludes the proof.

2 The limit of ideal MHD in astrophysical situations

(a) At the centre of the Sun, we have $T \sim 10^7$ K, and hence $\sigma \simeq 7 \times 10^{16}\,\mathrm{s}^{-1}$. If we take a typical scale length to be of the order of the Solar radius, then $L \sim 7 \times 10^{10}$ cm. Hence we get

$$t_d = \frac{4\pi\sigma L^2}{c^2} \simeq 5 \times 10^{18}\,\mathrm{s} \tag{50}$$

(b) For a typical Galactic HII region, $T \sim 10^4$ K and $L \sim 10$ pc. Therefore, we have $\sigma \sim 2 \times 10^{12}\,\mathrm{s}^{-1}$ and

$$t_d = \frac{4\pi\sigma L^2}{c^2} \simeq 3 \times 10^{31}\,\mathrm{s} \tag{51}$$

(c) At $z = 0$, the IGM is fully ionized, and has a temperature $T \sim 10^4$ K. A characteristic length scale is $L \simeq 1$ Mpc. Hence $\sigma \sim 2 \times 10^{12}\,\mathrm{s}^{-1}$ and

$$t_d = \frac{4\pi\sigma L^2}{c^2} \simeq 3 \times 10^{41}\,\mathrm{s} \tag{52}$$

Comparing with the age of the universe, $t_H \simeq 4 \times 10^{17}$ s, we see that flux freezing is a good approximation in all of these examples. It only starts to break down in systems where the fractional ionisation is very small, or when the phenomenon of **magnetic reconnection** takes place.

3 Amplification of magnetic fields

1. We have seen in the lectures that in the limit of ideal MHD the magnetic field obeys the following equation:

$$\frac{D}{Dt}\left(\frac{\mathbf{B}}{\rho}\right) = \left[\left(\frac{\mathbf{B}}{\rho}\right) \cdot \nabla\right]\mathbf{v} \tag{53}$$

The velocity field given in the text is divergence–free:

$$\nabla \cdot \mathbf{v} = \partial_x v_x + \partial_y v_y = 0 \tag{54}$$

Therefore, the continuity equation implies that if the density is uniform at $t = 0$, it will remain uniform at all times. Hence we can simplify ρ in (53). Expanding the convective derivative we obtain:

$$\partial_t \mathbf{B} + (\mathbf{v} \cdot \nabla)\mathbf{B} = (\mathbf{B} \cdot \nabla)\,\mathbf{v} \tag{55}$$

which written out in components and using the given velocity field is

$$\partial_t B_x + \alpha(x\partial_x - y\partial_y)B_x = \alpha B_x \tag{56}$$
$$\partial_t B_y + \alpha(x\partial_x - y\partial_y)B_y = -\alpha B_y \tag{57}$$
$$\partial_t B_z + \alpha(x\partial_x - y\partial_y)B_z = 0 \tag{58}$$

From the second and third equation, we see that if B_y and B_z are zero at $t = 0$, they will remain zero at all times. From the first equation, it is clear that if B_x is uniform at $t = 0$, it will remain uniform. Hence, the first equation reduces to

$$\partial_t B_x = \alpha B_x \tag{59}$$

The solution to this equation is a growing exponential, i.e.

$$B_x = B_0 \exp(\alpha t) \tag{60}$$

This result can be easily interpreted in terms of flux freezing. The field grows because the velocity field is such that two fluid elements are moving apart from each other, "stretching" the magnetic field that connects them.

2. Again the velocity field is divergence-free so we can start from (55). Expanding out in components we obtain

$$\partial_t B_x + v_y \partial_y B_x = 0 \tag{61}$$
$$\partial_t B_y + v_y \partial_y B_y = B_x(\partial_x v_y) \equiv B_x v_y' \tag{62}$$
$$\partial_t B_z + v_y \partial_y B_z = 0 \tag{63}$$

From the first equation we see that B_x remains constant at all times, since $\partial_y B_x = 0$ at $t = 0$. So $B_x(t) = B_0$. From the third equation we see that $B_z = 0$ at all times, since $B_z = 0$ everywhere at $t = 0$. From the second equation we see that B_y evolves as

$$B_y(t) = B_x v_y' t \tag{64}$$

i.e., it increases linearly with t with a rate proportional to the local amount of shear. With the particular form of v_y given in the text, we have

$$B_y(t) = \begin{cases} 0 \quad \text{if } x < -a \\ B_x(v_0/a)t \quad \text{if } -a \leq x \leq a \\ 0 \quad \text{if } x > a \end{cases} \tag{65}$$

Again, it is easy ti interpret this result in terms of flux freezing. Two fluid elements that are initially in the positions $(\pm a, y)$ will go in opposite direction, and the magnetic field that connects them "shears", stretching to remain connected to them.

3. We have seen in the lectures that in ideal MHD

$$\partial_t \left(\frac{1}{8\pi} B^2 \right) = -\boldsymbol{\nabla} \cdot \mathbf{S} + v_i \partial_j M_{ij} \tag{66}$$

where

$$M_{ij} = \frac{1}{4\pi} \left[B_i B_j - \frac{1}{2} B^2 \delta_{ij} \right] \tag{67}$$

and

$$\mathbf{S} = \frac{c}{4\pi} (\mathbf{E} \times \mathbf{B}) . \tag{68}$$

Integrating (66) over a fixed volume V bounded by a surface S and using the divergence theorem we obtain

$$\partial_t \left(\int_V \frac{1}{8\pi} B^2 \, \mathrm{d}V \right) = -\int_S \mathbf{S} \cdot \mathrm{d}\mathbf{s} + \int_V (v_i \partial_j M_{ij}) \, \mathrm{d}V \tag{69}$$

We now apply this general theorem to the results obtained in part 1 and 2 of the problem.

(1). We have

$$\mathbf{v} = \alpha x \hat{\mathbf{e}}_x - \alpha y \hat{\mathbf{e}}_y \tag{70}$$

and

$$\mathbf{B}(t) = B_0 \exp(\alpha t) \hat{\mathbf{e}}_x . \tag{71}$$

$\mathbf{B}$ is uniform at all times and its spatial derivatives vanish, so $\partial_j M_{ij}$=0. For the electric field we have

$$\mathbf{E} = -\frac{1}{c} \mathbf{v} \times \mathbf{B} \tag{72}$$

$$= -\frac{\alpha y B_0}{c} \exp(\alpha t) \hat{\mathbf{e}}_z \tag{73}$$

Hence the Poynting vector is

$$\mathbf{S} = \frac{c}{4\pi} (\mathbf{E} \times \mathbf{B}) \tag{74}$$

$$= -\frac{\alpha y B_0^2}{4\pi} \exp(2\alpha t) \hat{\mathbf{e}}_y \tag{75}$$

Now we want to calculate the various term of (69). Integrating the flux over the surface of a cube with edges at $\pm x_0$, $\pm y_0$, $\pm z_0$ only the two faces at $\pm y_0$ contribute, and these have an area of $A = (2x_0)(2y_0)$. We obtain

$$\int_S \mathbf{S} \cdot \mathrm{d}\mathbf{s} = -\frac{\alpha B_0}{4\pi} \exp(2\alpha t)(2y_0)(2x_0)(2z_0) \tag{76}$$

$$= -\frac{\alpha B_0}{4\pi} \exp(2\alpha t) V \tag{77}$$

$$\tag{78}$$

where V is the volume of the cube. According to the Poynting theorem above should be equal to minus the following:

$$\partial_t \left(\int_V \frac{1}{8\pi} B^2 \, \mathrm{d}V \right) = \partial_t \left(\int_V \frac{1}{8\pi} B_0^2 \exp(2\alpha t) \, \mathrm{d}V \right) = \frac{\alpha B_0}{4\pi} \exp(2\alpha t) V \tag{79}$$

and indeed it is. The term $v_i \partial_j M_{ij}$, which comes from $\mathbf{J} \cdot \mathbf{E}$, does not contribute. Hence, in this idealised example the energy is coming through the Poynting flux from “infinity”, which provides a continuous supply of fresh magnetic field which is eventually squashed and pushed towards the x axis, where it accumulates. The energy does not come through mechanical work from the fluid. However, if we cut the magnetic field at some height y_0, then there would be currents at this interface, and the fluid could transfer its kinetic energy to the magnetic energy through these currents.

(2). We have

$$\mathbf{v} = v_y(x) \hat{\mathbf{e}}_y \tag{80}$$

and

$$\mathbf{B} = B_x \, \hat{\mathbf{e}}_x + B_y \, \hat{\mathbf{e}}_y = B_0 \, \hat{\mathbf{e}}_x + B_0 v_y'(x) t \, \hat{\mathbf{e}}_y \tag{81}$$

The only non vanishing component of the velocity is $v_y \neq 0$, and the only non-vanishing spatial derivative of $\mathbf{B}$ is $\partial_x B_y \neq 0$. Hence the only contribution to $v_i \partial_j M_{ij}$ comes from the term $v_y \partial_x M_{xy}$, and we have

$$\begin{aligned} v_i \partial_j M_{ij} &= v_y \partial_x M_{xy} && (82) \\ &= v_y \partial_x \left[\frac{1}{4\pi} B_x B_y \right] && (83) \\ &= \frac{1}{4\pi} B_x v_y (\partial_x B_y) && (84) \\ &= \frac{B_0^2}{4\pi} v_y v_y'' t && (85) \end{aligned}$$

The Electric field is

$$\begin{aligned} \mathbf{E} &= -\frac{1}{c} \mathbf{v} \times \mathbf{B} && (86) \\ &= \frac{v_y B_0}{c} \hat{\mathbf{e}}_z && (87) \end{aligned}$$

Hence the Poynting vector is

$$\begin{aligned} \mathbf{S} &= \frac{c}{4\pi} (\mathbf{E} \times \mathbf{B}) && (88) \\ &= \frac{v_y B_0^2}{4\pi} \left[\hat{\mathbf{e}}_y - v_y' t \hat{\mathbf{e}}_x \right] && (89) \end{aligned}$$

Now consider (69) where the integral is over a cube $\pm x_0$, $\pm y_0$, $\pm z_0$. If $x_0 > a$, the Poynting flux integrated over all faces is zero. The flux on the surfaces at $\pm x_0$ is zero because $v_y' = 0$ there, and so $S_x = 0$ from (89). The flux on the surfaces at $\pm y_0$ is equal and opposite in sign, because S_y is identical on these two surfaces from (89), while the normal to the surface going out of the cube is in opposite directions. Hence if $x_0 > a$ we have

$$\int_S \mathbf{S} \cdot \mathrm{d}\mathbf{s} = 0. \tag{90}$$

Using (85) we have that

$$\int_V (v_i \partial_j M_{ij}) \, \mathrm{d}V = \int_V \left(\frac{B_0^2}{4\pi} v_y v_y'' t \right) \, \mathrm{d}V \tag{91}$$

with the form of v_y given in the text, v_y' is a step function, and v_y'' is the sum of two Dirac's delta functions:

$$v_y'' = [\delta(x+a) - \delta(x-a)] \, v_0/a \tag{92}$$

Therefore we have

$$\int_V (v_i \partial_j M_{ij}) \, \mathrm{d}V = \int_V \left(\frac{B_0^2 v_0^2 x t}{4\pi a^2} [\delta(x+a) - \delta(x-a)] \right) \, \mathrm{d}V \tag{93}$$

$$= \frac{B_0^2 v_0^2 t}{4\pi a^2} (2a)(2y_0)(2z_0) \tag{94}$$

The last term of (69) that we need to calculate is

$$\partial_t \left(\int_V \frac{1}{8\pi} B^2 \, \mathrm{d}V \right) = \partial_t \left(\int_V \frac{1}{8\pi} \left(B_x^2 + B_y^2 \right) \, \mathrm{d}V \right) \tag{95}$$

$$= \partial_t \left(\int_V \frac{1}{8\pi} B_0^2 (v_y')^2 t^2 \, \mathrm{d}V \right) \tag{96}$$

$$= \int_V \frac{1}{8\pi} B_0^2 (v_y')^2 2t \, \mathrm{d}V \tag{97}$$

$$= \frac{B_0^2 v_0^2 t}{4\pi a^2} (2a)(2y_0)(2x_0) \tag{98}$$

Putting together (90), (94) and (98) we see that (69) is satisfied. This time, the source of the energy is the work that the fluid does on the currents at $x = \pm a$. Indeed, by calculating $\boldsymbol{\nabla} \times \mathbf{B}$ it is easy to see that at $x = \pm a$ there are two sheet of currents at these positions. This energy is then transported via the Poynting flux to the inner parts, as can be seen by repeating the calculations above for a cube with $x_0 < a$.

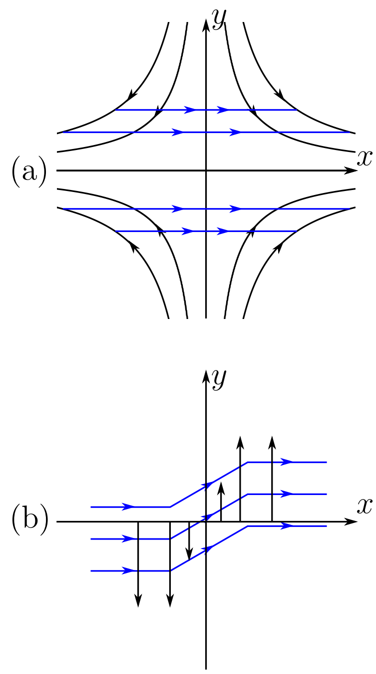

Figure 1: Sketch of the velocity field (black lines) and the magnetic field (blue lines) for (a) the first part of the problem (b) the second part of the problem.