Astrophysical Fluid Dynamics

Assignment #4: due May 30th

1 Analytical solutions of the Lane-Emden equation

The Lane-Emden equation with the boundary conditions $\theta(0) = 1$, $\theta'(0) = 0$ can be solved analytically only in the cases $n = 0$, $n = 1$ and $n = 5$.

(a) Find the analytical solution for $n = 0$.

(b) Find the analytical solution for $n = 1$.
Hint: substitute $\theta(\xi) = \zeta(\xi)/\xi$.

(c) Show that the analytical solution in the case $n = 5$ is

$$\theta = \frac{1}{\left(1 + \xi^2/3\right)^{1/2}} \tag{1}$$

2 Polytropic and isothermal slabs

We have studied the hydrostatic equilibria of polytropic and isothermal *spheres*. In this problem we want to carry out a similar study for polytropic and isothermal *slabs*. In other words, we consider infinitely extended sheets of gas piled up in hydrostatic equilibrium, supported by gas pressure and under their own self-gravity. Each sheet is infinitely extended and homogeneous in x and y. All quantities are assumed to be a function of z only, for example $\rho = \rho(z)$. The density distribution is assumed to be symmetric around $z = 0$, so that $\rho(z) = \rho(-z)$.

1. Find the analog of the Lane-Emden equation for a polytropic slab.
2. Find the analog of the isothermal Emden equation for an isothermal slab. Note that this equation can be integrated analytically. Find the corresponding density profile $\rho = \rho(z)$. Hint: you may find useful to know that for $x \geq 0$

$$\int \frac{\mathrm{d}x}{\left[1 - \exp(-x)\right]^{1/2}} = 2 \log\left(\sqrt{e^x - 1} + e^{x/2}\right) \tag{2}$$

3. Using a heuristic argument similar to the one provided in the lecture notes, discuss for what values of γ you would expect these slabs to be stable against gravitational collapse when motions are only along the z direction.

   IMPORTANT: note that even if you find that these slabs are stable when motions are solely in the z direction, it does *not* mean that they are stable when perturbations with motions in the x and y direction are allowed. In fact, they are not! It is possible for them to fragment into several pieces in the horizontal direction. The is similar to the ordinary Jeans instability.