Astrophysical Fluid Dynamics

Assignment #6: due June 20

1 Sound waves in a stratified atmosphere

Consider sound waves propagating in the vertical direction in an isothermal atmosphere with constant gravity $\mathbf{g} = -g\hat{\mathbf{z}}$. Assuming that everything is uniform in the x and y direction and that motions are only in the z direction, the equations of motion can be written as

$$\partial_t \rho + \partial_z (\rho v_z) = 0, \tag{1}$$

$$\partial_t v_z + v_z \partial_z v_z = -c_\mathrm{s}^2 \frac{\partial_z \rho}{\rho} - g, \tag{2}$$

where $c_\mathrm{s} =$ constant is the the sound speed and v_z is the velocity in the vertical direction.

(a) Show that if the atmosphere is in hydrostatic equilibrium, then

$$\rho(z) = \rho_0 e^{-z/H}, \tag{3}$$

$$v_\mathrm{z} = 0, \tag{4}$$

where ρ_0 is a constant and $H = c_\mathrm{s}^2/g$ is the vertical scale height of the atmosphere.

(b) Suppose that we now perturb the atmosphere away from this equilibrium solution using small density and velocity perturbations ρ_1 and v_{1z}. Derive the linearised equations of motions.

(c) Assume that ρ_1 and v_{1z} have the form

$$\rho_1 = \tilde{\rho}_1 \exp\left[i(kz - \omega t) - z/2H\right], \tag{5}$$

$$v_{1z} = \tilde{v}_{1z} \exp\left[i(kz - \omega t) + z/2H\right]. \tag{6}$$

Find the dispersion relation that relates ω and k.

(d) What happens when $\omega < c_\mathrm{s}/2H$? What does this imply for the propagation of very low frequency waves in the atmosphere?

2 Modes of a self-gravitating incompressible sphere

Consider a self-gravitating incompressible sphere of mass M, radius R and constant density. We want to find the modes of oscillation of this sphere, i.e. the possible frequencies of small waves travelling on its surface. Throughout this exercise, we use spherical coordinates (r, θ, ϕ).

(a) Show that the equilibrium state is

$$\rho_0 = \begin{cases} \frac{3M}{4\pi R^3} & \text{if} \quad r \leq R \\ 0 & \text{if} \quad r > R \end{cases} \tag{7}$$

$$P_0 = \begin{cases} \frac{3GM^2(R^2-r^2)}{8\pi R^6} & \text{if} \quad r \leq R \\ 0 & \text{if} \quad r > R \end{cases} \tag{8}$$

$$\Phi_0 = \begin{cases} -\frac{GM(3R^2-r^2)}{2R^3} & \text{if} \quad r \leq R \\ -\frac{GM}{r} & \text{if} \quad r > R \end{cases} \tag{9}$$

$$\mathbf{v}_0 = 0 \tag{10}$$

(b) Show that the linearised equations of motion around the equilibrium state can be written as

$$\partial_t \boldsymbol{\xi}_1 = \mathbf{v}_1(\mathbf{x}, t) \tag{11}$$

$$\partial_t \mathbf{v}_1 = -\frac{\boldsymbol{\nabla} P_1}{\rho_0} - \boldsymbol{\nabla}\Phi_1 \tag{12}$$

$$\boldsymbol{\nabla} \cdot \mathbf{v}_1 = 0 \tag{13}$$

$$\boldsymbol{\nabla}^2 \Phi_1 = \left(\xi_{1r}\omega_0^2\right) \delta(r - R) \tag{14}$$

where $\boldsymbol{\xi}_1$ is the lagrangian displacement, $\omega_0^2 \equiv 3GM/R^3$ is a characteristic frequency of the system, and δ is the Dirac delta.

(c) Assuming that all quantities vary in time as $e^{-i\omega t}$, show that the velocity can be expressed as the gradient of a scalar quantity U_1

$$\mathbf{v}_1 = \boldsymbol{\nabla} U_1 \tag{15}$$

and find U_1. Then show that for $r < R$ both U_1 and P_1 satisfy the Laplace equation:

$$\boldsymbol{\nabla}^2 U_1 = 0 \tag{16}$$

$$\boldsymbol{\nabla}^2 P_1 = 0 \tag{17}$$

(d) Find the appropriate boundary condition at $r = R$.

Hint: think of the water waves seen in the lecture.

(e) Show that the possible oscillation frequencies of the sphere are

$$\omega^2 = \frac{2l(l-1)}{3(2l+1)}\omega_0^2 \tag{18}$$

where $l = 0, 1, 2, \ldots$ is an integer.

Hint: recall that the general solution of the Laplace equation in spherical coordinates is

$$F(r, \theta, \phi) = \sum_{l=0}^{\infty} \sum_{m=-\infty}^{\infty} \left(A_l r^l + B_l r^{-l-1}\right) Y_l^m(\theta, \phi) \tag{19}$$

where A_l, B_l are constants and Y_l^m are the spherical harmonics.