Astrophysical Fluid Dynamics

Assignment #7: due December 11th

1. Phase velocities of MHD waves

Consider an isothermal fluid with a uniform density ρ. Assume that this fluid is permeated by a magnetic field with strength B_0 which is oriented parallel to the y axis. Suppose that an MHD wave is propagating in this fluid, with a wavevector $\mathbf{k}$ that lies in the $x - y$ plane, and let θ be the angle between $\mathbf{k}$ and $\mathbf{B}$.

(a) Assume that $v_{\rm A} > c_{\rm s}$. Plot the phase velocity of the MHD wave as a function of θ if it is (i) a fast magnetosonic wave; (ii) an Alfven wave; (iii) a slow magnetosonic wave. Be sure to indicate the values of the phase velocity at $\theta = 0$ and $\theta = 90°$. [*Note: plot all three lines on the same figure*].

(b) Repeat part (a), for the case where $v_{\rm A} < c_{\rm s}$.

2. The Prandtl-Meyer relation

(a) The shock jump conditions for normal shocks are

$$\rho_1 v_1 = \rho_2 v_2, \tag{1}$$

$$p_1 + \rho_1 v_1^2 = p_2 + \rho_2 v_2^2, \tag{2}$$

$$\frac{1}{2} v_1^2 + h_1 = \frac{1}{2} v_2^2 + h_2, \tag{3}$$

where h is the specific enthalpy, given by

$$h = \frac{\gamma}{\gamma - 1} \frac{p}{\rho}. \tag{4}$$

Use these to derive the **Prandtl-Meyer relation**

$$v_1 v_2 = c_*^2, \tag{5}$$

where c_*^2 is defined by the relation

$$\left(\frac{\gamma + 1}{\gamma - 1} \right) \frac{c_*^2}{2} = \frac{v^2}{2} + h, \tag{6}$$

and hence is conserved across a shock. [**Hint**: you will need to make use of all three jump conditions, plus the definition of h in terms of p and ρ. Note that you can assume that v_1 and v_2 are both greater than zero, and that it is sufficient to show that

$$(v_1 - v_2) \left(\frac{c_*^2}{v_1 v_2} - 1 \right) = 0, \tag{7}$$

as this implies that $v_1 v_2 = c_*^2$ provided that $v_1 \neq v_2$].

(b) c_* satisfies the inequality

$$c_2^2 > c_*^2 > c_1^2, \tag{8}$$

where c_1 and c_2 are the sound-speed in the upstream and downstream gas, respectively. Use this and the Prandtl-Meyer relation to show that for any shock, the upstream flow is supersonic ($v_1 > c_1$) and the downstream flow is subsonic ($v_2 < c_2$).