Astrophysical Fluid Dynamics

Assignment #10: due January 22nd

1. Stability of a rotating gas distribution

Suppose that two equal mass fluid elements A and B are located within an azimuthally symmetric rotating gas distribution. Let $R_{\rm A}$ and $R_{\rm B}$ be the distances from the rotation axis to fluid elements A and B, respectively. Assume that the angular velocity Ω varies as a function of radius within the disk. We now interchange the two fluid elements (so that A moves to $R_{\rm B}$ and B moves to $R_{\rm A}$) while ensuring that the specific angular momentum of each fluid element remains constant. How does the total kinetic energy of the system change? Under what circumstances will the energy decrease? What does this imply for the stability of the gas distribution?

2. Thermal instability

Consider a uniform gas distribution with density ρ_0, pressure p_0 and temperature T_0 that is initially at rest and that obeys the following fluid equations:

$$\frac{\partial \rho}{\partial t} + \nabla \cdot (\rho \vec{v}) = 0, \tag{1}$$

$$\frac{\partial \vec{v}}{\partial t} + \vec{v} \cdot \nabla \vec{v} = -\frac{1}{\rho} \nabla p, \tag{2}$$

$$\frac{\partial \epsilon}{\partial t} + \nabla \cdot (\epsilon \vec{v}) = -p \nabla \cdot \vec{v} - \rho \mathcal{L}. \tag{3}$$

Here, $\mathcal{L}$ is the cooling rate of the gas per unit volume, and $\epsilon = p/(\gamma - 1)$ is the internal energy density. We assume that the gas is initially in thermal equilibrium, and hence that $\mathcal{L}(\rho_0, T_0) = 0$. The gas is ideal and therefore also obeys the ideal gas law

$$p = \frac{R}{\mu} \rho T, \tag{4}$$

where $R \equiv k_{\rm b}/m_{\rm H}$ is the specific gas constant and μ is the mean molecular weight. Suppose now that we perturb this gas distribution with perturbations of the form

$$a(\vec{x}, t) = a_1 \exp\left(\omega t + i\vec{k} \cdot \vec{x}\right). \tag{5}$$

[**Note** that with perturbations of this form, imaginary ω corresponds to an oscillatory perturbation, while real, positive ω corresponds to a growing mode.]

(a) Show that to first order, the behaviour of the perturbations is governed by the following set of equations:

$$\begin{aligned} \frac{\partial \rho_1}{\partial t} + \rho_0 \nabla \cdot \vec{v}_1 &= 0, & (6) \\ \rho_0 \frac{\partial \vec{v}_1}{\partial t} &= -\nabla p_1, & (7) \\ \frac{1}{\gamma - 1} \frac{\partial p_1}{\partial t} + \frac{\gamma}{\gamma - 1} p_0 \nabla \cdot \vec{v}_1 &= -\rho_0 \rho_1 \mathcal{L}_\rho - \rho_0 T_1 \mathcal{L}_T, & (8) \\ \frac{p_1}{p_0} - \frac{\rho_1}{\rho_0} - \frac{T_1}{T_0} &= 0, & (9) \end{aligned}$$

where $\mathcal{L}_\rho \equiv (\partial \mathcal{L}/\partial \rho)_T$ and $\mathcal{L}_T \equiv (\partial \mathcal{L}/\partial T)_\rho$.

(b) Using these equations, show that the dispersion relation of the perturbations can be written as

$$\omega^3 + c_{\rm s} k_T \omega^2 + c_{\rm s}^2 k^2 \omega + \frac{c_{\rm s}^3 k^2}{\gamma} (k_T - k_\rho) = 0, \quad (10)$$

where $c_{\rm s} = (\gamma p_0/\rho_0)^{1/2}$, and

$$\begin{aligned} k_\rho &= \frac{\mu(\gamma - 1)\rho_0 \mathcal{L}_\rho}{R c_{\rm s} T_0}, & (11) \\ k_T &= \frac{\mu(\gamma - 1)\mathcal{L}_T}{R c_{\rm s}}. & (12) \end{aligned}$$

(c) Show that in the limit $k \to 0$, the perturbations are unstable only if $\mathcal{L}_T < 0$.