Stellar Structure & Evolutions

Literature:

- Steven S. Shore: $\boxed{\text{The tapestry of modern astrophysics}}$

- Bradley W. Carroll, Dale A. Ostlie: $\boxed{\text{In introduction to modern astrophysics}}$

  → web info (such as stellar structure code STATSTAR):
  http://www.aw-bc.com/astrophysics

- Peter Bodenheimer, Greg P. Laughlin, Michał Różyczka, Hal W. Yorke: $\boxed{\text{Numerical Methods in Astrophysics}}$

  → contains CD with code:
  STELLAR, a Lagrangian stellar evolution code using the Henyey method.

Stellar Structure & Evolution

We seek structure of star on main sequence.

1. Introduction

$\boxed{\text{Stars are huge self-gravitating spheres of gas}}$

- stars are very massive
  ↳ they must reach very high internal temperaturse to generate sufficient pressure to guarantee stability against gravitational collapso

- stars have no walls
  ↳ they must have outer boundaries which communicate all excess energy from their interior to space (they radiate energy away)

↳ in the end, this means, they never are in a true equilibrium state.
(however, we often assume quasi-equi-brium)

Stars are huge self-gravitating spheres of gas, with nuclear reactions providing the energy source for quasi-equilibrium

↳ Structure & evolution governed by equations of hydrodynamics

Our goal here: Understand structure on main sequence; i.e. structure of star in quasi-equilibrium;

↳ We seek time-independent solutions of hydro equations

Fundamental assumption: stars are perfect spheres

↳ we consider simple, 1D spherical symmetry

↳ this simplifies our equations.

Equations of hydrodynamics

[recall from theoretical astrophysics course. derived from Boltzmann equation via moment-building.]

continuity of mass

$$\frac{d\rho}{dt} = \frac{\partial \rho}{\partial t} + (\vec{v}\cdot\vec{\nabla})\rho = -\rho\,\vec{\nabla}\cdot\vec{v}$$

Navier-Stokes: momentum equation

$$\frac{d\vec{v}}{dt} = \frac{\partial \vec{v}}{\partial t} + (\vec{v}\cdot\vec{\nabla})\vec{v} = -\frac{1}{\rho}\vec{\nabla}P - \vec{\nabla}\phi$$
$$+\eta\vec{\nabla}^2\vec{v} + \left(\zeta + \frac{\eta}{3}\right)\vec{\nabla}\cdot(\vec{\nabla}\cdot\vec{v})$$

energy equation

$$\frac{d\epsilon}{dt} = \frac{\partial \epsilon}{\partial t} + (\vec{v}\cdot\vec{\nabla})\epsilon = T\frac{ds}{dt} - \frac{P}{\rho}\vec{\nabla}\cdot\vec{v}$$
$$= T\frac{ds}{dt} - \frac{P}{\rho^2}\frac{d\rho}{dt}$$
$$= T\frac{ds}{dt} - P\frac{dV}{dt}$$

Poisson equation

$$\vec{\nabla}^2\phi = 4\pi G\rho$$

Closure equation $\longrightarrow$ $\boxed{\text{equation of state}}$

$$P = \mathrm{fct}(\rho, T)$$

EOS of ideal gas: $\boxed{PV = NkT}$

$$\hookrightarrow \quad P = \frac{k\rho T}{\mu m_p}$$

with $k = 1{,}38 \cdot 10^{-16}$ erg/K

& μ = mean molecular weight

m_p = mean mass of proton $= 1{,}67 \cdot 10^{-24}$ g

[be careful, this changes throughout star]

Polytropic EOS: $\boxed{P = K \cdot \rho^{\gamma}}$

$$\gamma = \text{polytropic index} = \frac{\text{specific heat at constant } P}{\text{specific heat at const. } V}$$

$$\boxed{\gamma = \frac{c_p}{c_V}} \; ; \; \boxed{c_p = c_V + R}$$

for monoatomic gas: $\boxed{\gamma = \frac{5/2\,R}{3/2\,R} = \frac{5}{3}}$

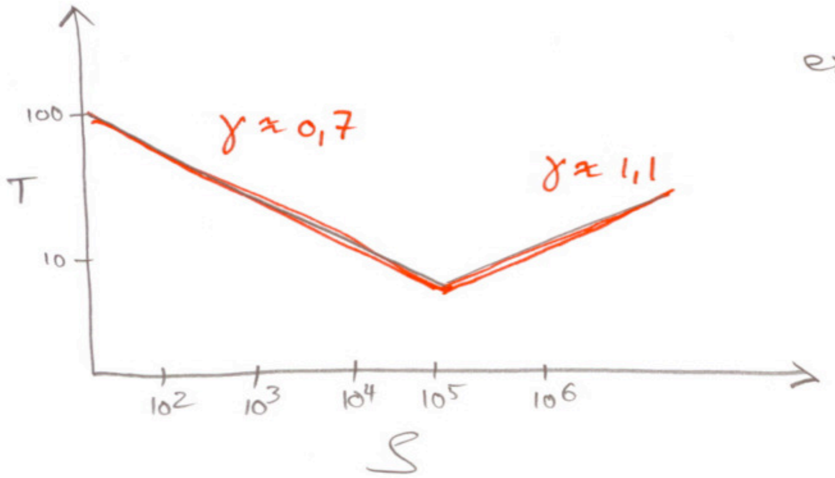
isothermal EOS: $\boxed{P = K \cdot \rho = c_s^2 \rho}$

isothermal $\gamma = 1$

in general: thermodynamic state of the gas depends on balance between heating and cooling processes

↳ the effective γ can have any value > 0 !

in general: $\boxed{\gamma = 1 + \frac{d \ln T}{d \ln \rho}}$



example: ISM

gas cools more strongly as it gets compressed

$\Lambda \propto \rho^2$

↳ $T \downarrow$

coupling to dust → gas heats up again

↳ $T \uparrow$

Our goal is to determine, how

ρ, P, T, ϵ

change with position in star.

- Fundamental assumption: $\boxed{\text{stars are spheres}}$

↳ <u>1D spherical symmetry</u>

↳ $(x, y, z) \longrightarrow (r, \theta, \phi)$

$$\left.\begin{aligned} \vec{\nabla}_{\vec{x}} &= \vec{e}_x \partial_x + \vec{e}_y \partial_y + \vec{e}_z \partial_z \\ \vec{\nabla}_{\vec{r}} &= \vec{e}_r \frac{\partial}{\partial r} + \frac{\vec{e}_\theta}{r} \frac{\partial}{\partial \theta} + \frac{\vec{e}_\phi}{r \sin\theta} \frac{\partial}{\partial \phi} \end{aligned}\right\} \text{spherical coordinates}$$

spherical symmetry: $\frac{\partial}{\partial \theta} \to 0 \quad \frac{\partial}{\partial \phi} \to 0$

↳ only <u>radial</u> dependency remains.

We seek all "interesting" variables as function of

Radius r

$$\boxed{\frac{dP}{dr} = ? \qquad \frac{dM_r}{dr} = ? \qquad \frac{dL_r}{dr} = ? \qquad \frac{dT}{dr} = ?}$$

or equivalently (as there is a one-to-one relation between r and M_r) as function of

Mass M_r

$$\boxed{\frac{dP}{dM_r} = ? \qquad \frac{dr}{dM_r} = ? \qquad \frac{dL_r}{dM_r} = ? \qquad \frac{dT}{dM_r} = ?}$$

r = Radius from center outwards

M_r = Mass within radius r

P = pressure

L = luminosity

T = temperature

2. Hydrostatic Balance

- Consider star in quasi-static equilibrium
  ↳ no acceleration

- in Navier-Stokes: $\frac{d\vec{v}}{dt} = 0$

↳ $\frac{d\vec{v}}{dt} = 0 = -\frac{1}{\rho}\vec{\nabla}P - \vec{\nabla}\Phi$

↳ $$\boxed{\frac{dP}{dr} = -\rho\frac{GM(r)}{r^2}} \quad (I)$$

$\left.\begin{array}{l}\text{force due}\\ \text{to pressure}\\ \text{gradient}\end{array}\right\} = \left\{\begin{array}{l}\text{force due}\\ \text{to gravity}\end{array}\right.$

we use $\vec{\nabla}\phi = \frac{d\phi}{dr} = \frac{d}{dr}\left[-\frac{GM(r)}{r}\right] = \frac{GM(r)}{r^2}$

- Enclosed mass or: equivalence of r and $M(r)$:

$$\boxed{\begin{aligned} M_r = M(r) &= \int_0^r 4\pi\rho(r')r'^2dr' \\ &= \text{mass interior to } r\end{aligned}}$$

↳ $\boxed{\frac{dM_r}{dr} = 4\pi\rho(r)r^2}$ (II) ⟷ $\boxed{\frac{dr}{dM_r} = \frac{1}{4\pi\rho r^2}}$

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using mass as independent variable

$$\boxed{\frac{dP}{dr} = -\rho(r)\,\frac{GM(r)}{r^2}} \longleftrightarrow \boxed{\frac{dP}{dM_r} = -\frac{GM_r}{4\pi r^4}}$$

→ from (1) follows: star must be hot in the center in order to hold it up against contraction by sufficient pressure gradients.

as first approach say $\frac{dP}{dr} \to \frac{P}{R}$ (R = radius of $*$)
and assume ideal gas law $P = \frac{\rho k T}{\mu m_p}$

↳ $\frac{P}{R} \approx \bar{\rho}\,\frac{GM}{R^2}$

$$\frac{1}{\cancel{R}}\,\frac{\cancel{\rho}\,kT_c}{\mu m_p} \approx \cancel{\bar{\rho}}\,\frac{GM}{R^{\cancel{2}}} \longrightarrow \boxed{kT_c \approx \mu\,\frac{GM m_p}{R}}$$

↳ if star shrinks, it must get hotter for fixed M $\boxed{R\downarrow \quad T_c\uparrow}$

↳ if mean molecular weight increases,* star must get hotter in center for fixed M & R $\boxed{\mu\uparrow \quad T_c\uparrow}$

[important for stellar evolution as H burns into He]

3. Dynamical Time

- So far we assumed quasi-equilibrium, i.e. $\frac{d\vec{v}}{dt} = 0$

  What happens, if this is not the case?

- $\hookrightarrow$ Navier-Stokes:

$$\frac{d\vec{v}}{dt} = -\underbrace{\cancel{\frac{1}{\rho}\vec{\nabla}P}}_{\text{drop pressure force}} - \frac{GM}{r^2}\,\vec{e}_r$$

$\hookrightarrow$ $$\frac{d\vec{v}}{dt} = -\frac{GM(r)}{r^2}\,\vec{e}_r$$

- Collapse on a dynamical time scale:

$$\boxed{v \approx \frac{R}{\tau_{dyn}}} \qquad \boxed{M(R) \approx \bar{\rho} R^3} \qquad \boxed{\frac{d}{dt} \approx \frac{1}{\tau_{dyn}}}$$

$\hookrightarrow$ $$\frac{R}{\tau_{dyn}^2} \approx \frac{G\bar{\rho}R^3}{R^2}$$

$\hookrightarrow$ $$\tau_{dyn} \approx \frac{1}{\sqrt{G\rho}} \approx \frac{1\ \text{hour}}{(\rho / 1\,\text{g}\,\text{cm}^{-3})^{1/2}}$$

- this is the shortest time scale available to the star.

4. Virial Balance

- Instead of looking at the balance of forces, we can compare energies

$\hookrightarrow$ $\boxed{\text{Virial Theorem}}$

Here we derive it by integrating equation (1):

- Take $\frac{dP}{dr} = -\rho \frac{GM(r)}{r^2}$ and multiply by $4\pi r^3 dr$ on both sides and integrate:

$$\int_0^R 4\pi r^3 \frac{dP}{dr}\, dr = -\int_0^R \frac{GM(r)}{r} \cdot \underbrace{4\pi \rho r^2 dr}_{dM_r}$$

$$= -\int_0^R \frac{GM_r}{r}\, dM_r = +W$$

LHS: integrate by parts:

$$\int_0^R 4\pi r^3 \frac{dP}{dr}\, dr = \underbrace{4\pi r^3 P(r)\Big|_0^R}_{=0} - 3\int_0^R P(r) \cdot \underbrace{4\pi r^2 dr}_{dV}$$

$$\begin{bmatrix} \text{at } R: P(R) = 0 \\ 0: R = 0 \end{bmatrix}$$

$$= -3 \int_0^R P\, dV \overset{!}{=} -2U$$

- From the theoretical astrophysics lecture we know, the "full" virial theorem for self-grav-gas is

$$\boxed{\frac{1}{2}\ddot{I} = 2T + 2U + W}$$

in equilibrium $\ddot{I} = 0$. (tensor of inertia)

- If there is no bulk motion or rotation, $T = 0$

$\hookrightarrow$ $\boxed{2U + W = 0}$ $\quad * \; U = \frac{3}{2}\int P dV$

$* \; W = \frac{1}{2}\int \Phi \rho dV$

- Note: $\boxed{U = \frac{3}{2}\int P dV \text{ is } \underline{\text{NOT}} \text{ the internal energy } U_{int} = \frac{3}{2} NkT \;!}$

not all the internal energy is able to do work !

- internal energy density: $\epsilon = \frac{3}{2} nkT$; Pressure $P = nkT$
- we know: $\gamma \cdot \epsilon = \frac{5}{3} \cdot \frac{3}{2} nkT = \frac{5}{2} nkT = \left(\frac{3}{2} + 1\right) nkT$

$= \epsilon + P$

with $\gamma = \frac{c_p}{c_V}$ $\quad \hookrightarrow \boxed{P = (\gamma - 1)\epsilon}$

thus: $\boxed{U = \frac{3}{2}\int P\,dV} = \frac{3}{2}\int(\gamma-1)\,\epsilon\, dV$

$$= \frac{3}{2}(\gamma-1)\underbrace{\int \epsilon\, dV}_{\text{internal energy } U_{int}}$$

$$\boxed{= \frac{3}{2}(\gamma-1)\,U_{int}}$$

↳ virial theorem: $2U + W = 0$

$$\boxed{3(\gamma-1)U_{int} + W = 0}$$

• total energy: $\boxed{E_{tot} = U_{int} + W}$

↳ together:

$$E_{tot} = U_{int} - 3(\gamma-1)\,U_{int} = -(3\gamma-4)\,U_{int}$$

because $U_{int} > 0$, the equilibrium system is bound, i.e. $E < 0$, only if

$$\boxed{\gamma > \frac{4}{3}}$$

this defines a critical polytrope for

What happens as $\gamma \to 4/3$?

$\to$ consider $\ddot{I} = 2U + W$

$$= 3(\gamma-1)U_{int} + W$$

$$= 3(\gamma-1)(E-W) + W$$

$$= 3(\gamma-1)E - (3\gamma-4)W$$

$$\to \quad E = \frac{\gamma - 4/3}{\gamma-1} W + \frac{1}{3(\gamma-1)} \ddot{I}$$

for $\gamma \to 4/3$ the first term on RHS vanishes and if we manage to keep $E<0$ then $\ddot{I} < 0$; implying contraction.

[Also seen via linear stability analysis (CO § 14.3) for $\gamma < 4/3$ exponentially growing modes $\to$ collapse.]

Application of Virial Theorem:

Kelvin - Helmholtz timescale

- for a $\star$, the total energy is always negative

$$\boxed{E_{tot} = -(3\gamma-4)\,U_{int}} \quad \& \quad \boxed{E_{tot} = \frac{3\gamma-4}{3\gamma-3}\cdot W}$$

if $\gamma > 4/3$

$$W = -\int_0^R \frac{GM(r)}{r}\,4\pi\rho(r)\,r^2 dr =$$

assume constant, mean density $\bar{\rho} = \frac{M}{\frac{4\pi}{3}R^3}$

$$\approx -\int_0^R G\,\overbrace{\tfrac{4\pi}{3}\bar{\rho}r^3}^{M(r)}\cdot 4\pi\bar{\rho}\cdot r\,dr$$

$$= -\frac{16\pi^2}{15}G\bar{\rho}^2R^5 = \underline{-\frac{3}{5}\frac{GM^2}{R}}$$

- for $\gamma = \frac{5}{3}$: $\quad E_{tot} = \frac{5-4}{5-3}W = \frac{1}{2}W = \underline{-\frac{3}{10}\frac{GM^2}{R}}$

- a star that loses (internal) energy must <u>contract</u>

$$\boxed{E_{tot} \text{ gets smaller (more negative) if } R \text{ gets smaller!}}$$

- luminosity of star

$$L = -\frac{dE}{dt} = +\frac{3}{10}\frac{GM^2}{R}\frac{\dot{R}}{R}$$

$\Rightarrow$ <u>constant</u> luminosity requires <u>constant</u> contraction $\dot{R}$!

- characteristic timescale

$$\dot{R} = \frac{R}{t_{KH}} \rightarrow \boxed{t_{KH} = \frac{3}{10} \frac{GM^2}{RL}}$$

this is the Kelvin – Helmholtz – timescale

for the Sun:

$$L_\odot = 4 \cdot 10^{33} \ \mathrm{erg/s} = 4 \cdot 10^{33} \ \frac{\mathrm{g\,cm^2}}{\mathrm{s^3}}$$

$$M_\odot = 2 \cdot 10^{33} \ \mathrm{g}$$

$$G = 6{,}67 \cdot 10^{-8} \ \frac{\mathrm{cm^3}}{\mathrm{g\,s^2}}$$

$$R_\odot = 7 \cdot 10^{10} \ \mathrm{cm}$$

$$\hookrightarrow \quad t_{KH} = \frac{3}{10} \cdot \frac{6{,}67 \cdot 10^{-8} \cdot 4 \cdot 10^{66} \, \cancel{\mathrm{g^2}} \cdot \frac{\cancel{\mathrm{cm^3}}}{\cancel{\mathrm{g}}\,\cancel{\mathrm{s^2}}}}{7 \cdot 10^{10} \, \cancel{\mathrm{cm}} \cdot 4 \cdot 10^{33} \, \cancel{\mathrm{g}} \, \cancel{\mathrm{cm^2}}/\mathrm{s}^{\cancel{3}}}$$

$$= 9{,}5 \cdot 10^{14} \ \mathrm{s} = 30 \ \mathrm{Myr}$$

$\hookrightarrow$ this timescale is short compared to age of Earth. $\Rightarrow$ grav. contraction cannot be energy source of the Sun!

- Note: $\boxed{\text{star is always in hydrostatic balance as it contracts, because } t_{KH} \gg \tau_{dyn} \approx 1 \text{ hour}}$

- a star radiates, i.e. loses energy continuously

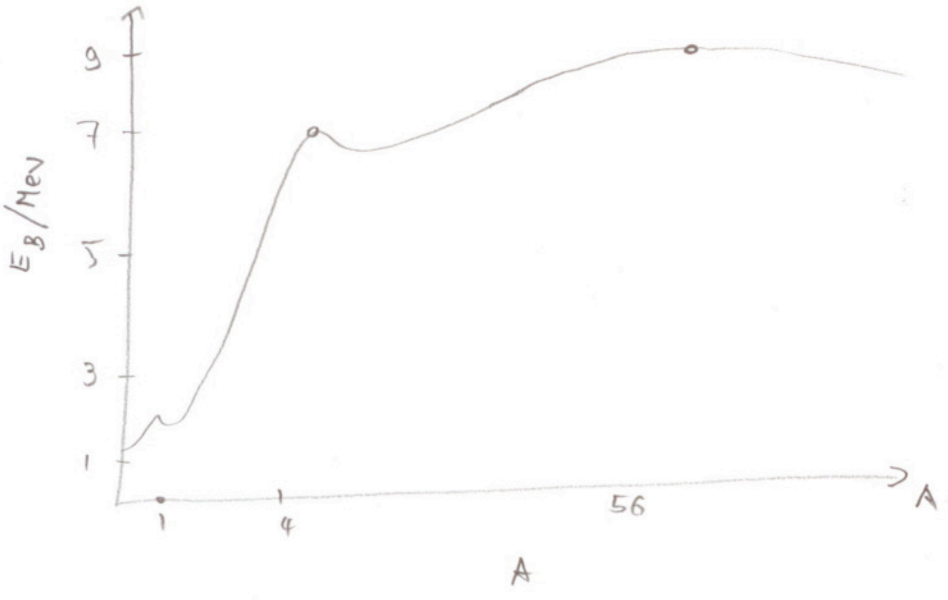
  $\rightarrow$ it can only escape contraction if a new source of energy becomes available to compensate for the radiative losses at surface.

- What can that be?

$$E_{grav} \approx \frac{GM^2}{R} \rightarrow \frac{E_{grav}}{M} = \frac{GM}{R} \approx 10^{15} \frac{erg}{g}$$

chemistry: $\frac{5\,eV}{atom} \approx 5 \cdot 10^{12} \frac{erg}{g}$

nuclear reactions: $\frac{7\,MeV}{m_p} \approx 7 \cdot 10^{18} \frac{erg}{g}$



5. Equation of state for radiation-dominated gases

- Recall: polytropic EOS: $P = K \cdot \rho^{\gamma}$

- recall also: ideal gas law: $PV = NkT$
$$= \frac{R}{\mu} T$$

- recall first law of TD:

$$dQ = dU + P\,dV$$

dQ = heat change
dU = change of int. energy

$U = u \cdot V$ energy density for radiation is $u = \frac{4\pi}{c} \int B_\nu(T)\, d\nu$

$P = \frac{1}{3} u$

- entropy change: $dS = \frac{dQ}{T}$

$$\hookrightarrow dS = \frac{V}{T} \frac{du}{dT} dT + \frac{u}{T} dV + \frac{1}{3} \frac{u}{T} dV$$
$$= \frac{V}{T} \frac{du}{dT} dT + \frac{4}{3} \frac{u}{T} dV$$

build $\left.\frac{\partial S}{\partial T}\right|_V = \frac{V}{T} \frac{du}{dT}$ & $\left.\frac{\partial S}{\partial V}\right|_T = \frac{4}{3} \frac{u}{T}$

build $\frac{\partial^2 S}{\partial V \partial T} = \frac{1}{T} \frac{du}{dT}$ & $\frac{\partial^2 S}{\partial T \partial V} = -\frac{4}{3} \frac{u}{T^2} + \frac{4}{3} \frac{1}{T} \frac{du}{dT}$

because $\frac{\partial^2 S}{\partial V \partial T} = \frac{\partial^2 S}{\partial T \partial V} \rightarrow \frac{1}{3}\frac{1}{T}\frac{du}{dT} = \frac{4}{3}\frac{u}{T^2}$

$\hookrightarrow \frac{du}{u} = 4\frac{dT}{T} \rightarrow \ln u = 4 \ln T + \ln a$

with $\ln a$ = const. of integration

$\hookrightarrow$ Stefan-Boltzmann law: $\boxed{u(T) = a \cdot T^4}$

$\hookrightarrow$ Radiation pressure: $\boxed{P_{rad} = \frac{1}{3} u = \frac{a}{3} T^4}$

the total energy of the radiation-influenced gas is

$$\boxed{U = \frac{3}{2}\frac{R}{\mu} T + aT^4 V = c_V T + aT^4 V}$$

total pressure:

$$\boxed{P = \frac{R}{\mu}\frac{T}{V} + \frac{a}{3}T^4} = P_{gas} + P_{rad}$$

Now define the ratio of gas pressure to total pressure: $\boxed{\beta = \frac{P_g}{P}}$

with $0 \leq \beta \leq 1$.

then $\boxed{P_{gas} = \frac{\rho k T}{\mu m_p} = \beta P}$ ⓐ for gaspressure

and $\boxed{P_{rad} = \frac{a}{3} T^4 = (1-\beta) P}$ ⓑ for radiation press

We are looking for a polytropic EOS that is independent of T, we combine both expressions to eliminate T:

- in ⓐ: $T = \frac{\beta P}{\rho k} \cdot \mu m_p$ insert in ⓑ:

$$\longrightarrow \quad \frac{a}{3} \left(\frac{\beta P \mu m_p}{\rho k} \right)^4 = (1-\beta) P$$

- then: $\boxed{P = K \cdot \rho^{4/3}}$

with $K = \left(\frac{3(1-\beta)}{a} \right)^{1/3} \left(\frac{k}{\beta \mu m_p} \right)^{4/3}$

- The EOS of radiation dominated systems has a polytropic index $\boxed{\gamma = 4/3}$
  (this corresponds to a polytrop of index $n = \frac{1}{\gamma - 1}$)

- This is similar to relativistic, degenerate objects (clear, because photons are relativistic)

- Also radiation dominated stars are close to the stability limit !

We can estimate, when P_{rad} dominates over P_{gas}:

- recall from hydrostatic balance:

$$kT \sim \frac{GMm_p}{R} \qquad \text{set } \mu = 1$$

- we have $\frac{P_{rad}}{P_{gas}} \sim \frac{aT^4 m_p}{\rho kT}$

we know (theoretical astrophysics):

$$a = \frac{8\pi^5}{15}\frac{k^4}{c^3h^3} = 7{,}55\cdot 10^{-15}\,\frac{\text{erg}}{\text{cm}^3\text{K}^{-4}}$$

$$\approx \frac{k^4}{(\hbar c)^3}$$

$\rightarrow$ $$\frac{P_{rad}}{P_{gas}} \sim \frac{k^{\cancel{4}\,3}}{(\hbar c)^3}\cdot\frac{T^{\cancel{4}\,3} m_p}{\rho \cancel{kT}} = \frac{(kT)^3}{(\hbar c)^3}\frac{m_p}{\rho} = \left(\frac{GMm_p}{\cancel{R}}\right)^3 \frac{\cancel{R^3}}{M}\cdot\frac{m_p}{(\hbar c)^3}$$

independent of R!

$$\sim \underbrace{\left(\frac{Gm_p^2}{\hbar c}\right)^3}_{\alpha_G = \frac{Gm_p^2}{\hbar c} = 6\cdot 10^{-39}} \left(\frac{M}{m_p}\right)^2$$

$$\sim \alpha_G^2 \left(\frac{M}{m_p}\right)^2$$

only function of mass

$\rightarrow$ critical mass: $M \approx \alpha_G^{-3/2}\, m_{proton}$

$$\approx 2\cdot 10^{57}\, m_p$$

$$\approx 4 \cdot M_\odot$$

[Real Answer: $M \gtrsim 60\, M_\odot$ or so $\rightarrow$ we dropped too many prefactors]