5. Equation of State for degenerate gas

- for non-relativistic particles: $\boxed{P = K\rho^{5/3}}$
- for relativistic particles $\boxed{P = K'\rho^{4/3}}$

Derivation a la Padmanabhan, p. 2+3:

$$P = \frac{1}{3}\int_0^\infty n(\epsilon)p(\epsilon)v(\epsilon)\,d\epsilon$$

- rate of momentum transfer from particles with energy ϵ
- $\frac{1}{3}$ comes from isotropy

$$p = \gamma m v$$

$$\epsilon = (\gamma - 1)mc^2$$

$$\gamma = \left(1 - \frac{v^2}{c^2}\right)^{-1/2}$$

- the system is called ideal, when the kinetic energy dominates over their interaction energy

↳ then: $$P = \frac{1}{3}\int_0^\infty n\epsilon\left(1 + \frac{2mc^2}{\epsilon}\right)\left(1 + \frac{mc^2}{c}\right)^{-1} d\epsilon$$

Non-relativistic limit: $mc^2 \gg \epsilon$

$$P \approx \frac{2}{3}\int_0^\infty n\epsilon\,d\epsilon = \frac{2}{3}\langle n\epsilon\rangle = \frac{2}{3}u$$

Rel. limit: $mc^2 \ll \epsilon$ (or particles are massless: photons)

$$P \approx \frac{1}{3}\int_0^\infty n\epsilon\,d\epsilon = \frac{1}{3}\langle n\epsilon\rangle = \frac{1}{3}u$$

with u being energy density

- kinetic energy ϵ for particle with momentum p and mass m:

$$\epsilon = \sqrt{p^2c^2 + m^2c^4} - mc^2 = \begin{cases} \frac{p^2}{2m} & p \ll mc \\ pc & p \gg mc \end{cases}$$

$$[p] = \frac{g\,cm}{s}$$

for <u>DEGENERATE GAS</u>:

[Pauli + Heisenberg]

- uncertainty principle: gives minimum volume a free particle can occupy in phase space $\Delta x \Delta p = h$

- if particle separation, on average, given by number density, $\Delta x = n^{-1/3}$, then $p_F = h \cdot n^{1/3}$. [exclusion principle!]

- Because for isotropic gas

$$\boxed{P_{degenerate} = \frac{2}{3} u_F = \frac{2}{3} \epsilon \rho \quad NR \\ = \frac{1}{3} u_F = \frac{1}{3} \epsilon \rho \quad Rel.}$$

we can write in

NON-REL. case $\quad P_{deg} \propto \frac{p_F^2 n}{m}$

$$\hookrightarrow \boxed{P_{deg}^{NR} = K^{NR} \rho^{5/3}}$$

REL. case $\quad P_{deg} \propto p_F n$

$$\hookrightarrow \boxed{P_{deg}^{ER} = K^{ER} \rho^{4/3}}$$

- for degenerate relativistic particles that are compressed, the pressure increase is smaller since also the <u>"apparent" mass</u> is increased the effect is as if the mean molecular weight is increased, the medium becomes more compressible ! $\boxed{\text{EOS becomes softer}}$

Note the fully relativistic degenerate case has the same scaling as radiation EOS !

6. Structure of Polytropic Spheres

- We go back to the equations of hydrostatic balance:

$$\boxed{\begin{array}{c} \dfrac{dP}{dr} = -\rho \dfrac{GM(r)}{r^2} \\ \hline \dfrac{dM(r)}{dr} = 4\pi \rho r^2 \end{array}} \quad \begin{array}{c} \text{(I)} \\ \\ \text{(II)} \end{array}$$

These are the first of the desired stellar structure equations.

We note, the system can be solved <u>without</u> reference to any energy or energy transport equation!

- We take the radial derivative of (I):

$$\frac{d}{dr}\left(\frac{r^2}{\rho}\frac{dP}{dr}\right) = -G\frac{dM_r}{dr}$$

and use (II) for the RHS

$$\frac{d}{dr}\left(\frac{r^2}{\rho}\frac{dP}{dr}\right) = -G\, 4\pi\rho r^2$$

or

$$\boxed{\frac{1}{r^2}\frac{d}{dr}\left(\frac{r^2}{\rho}\frac{dP}{dr}\right) = -4\pi G\rho} \quad (*)$$

this is a Poisson equation

Because in hydrostatic equilibrium pressure gradients follow grav. potential gradients: $\frac{1}{\rho}\frac{dP}{dr} = -\frac{d\phi}{dr}$

we can also write:

$$\boxed{\frac{1}{r^2}\frac{d}{dr}\left(r^2\frac{d\phi}{dr}\right) = +4\pi G\rho}$$

we solve (*) by using a polytropic EOS:

$$\boxed{P = K\rho^{\gamma}}$$

↳ we get $$\boxed{\frac{\gamma K}{r^2}\frac{d}{dr}\left[r^2\rho^{\gamma-2}\frac{d\rho}{dr}\right] = -4\pi G\rho}$$

- Often people use a different notation:
  define $\gamma = \frac{n+1}{n}$ where <u>n = polytropic index</u>

↳ $$\left(\frac{n+1}{n}\right)\frac{K}{r^2}\frac{d}{dr}\left[r^2\rho^{\frac{1-n}{n}}\frac{d\rho}{dr}\right] = -4\pi G\rho$$

- we simplify by expressing density in terms of a scaling factor: <u>central density ρ_c</u> and a dimensionless function $D_n(r)$:

$$\boxed{\rho(r) = \rho_c\cdot\left[D_n(r)\right]^n \quad \text{with } 0\le D_n(r)\le 1}$$

we get:

$$(n+1)\left(\frac{K\rho_c^{\frac{1-n}{n}}}{4\pi G}\right)\frac{1}{r^2}\frac{d}{dr}\left[r^2\frac{dD_n(r)}{dr}\right] = -D_n^n(r)$$

- simplify further:

$$\boxed{\lambda_n = \left[(n+1)\left(\frac{K\rho_c^{\frac{1-n}{n}}}{4\pi G}\right)\right]^{1/2}}$$

and introduce dimensionless variable ξ

$$\boxed{r = \lambda_n\cdot\xi}$$

and we finally get

$$\boxed{\frac{1}{\xi^2}\frac{d}{d\xi}\left(\xi^2 \frac{dD_n}{d\xi}\right) = -D_n^n}$$ (LE)

Lane Emden equation

- Solving (LE) for the dimensionless fct. $D_n(\xi)$ in terms of ξ for a specific index n leads directly to the density profile $\rho_n(r)$. The polytropic EOS $P_n(r) = K\rho_n^{(n+1)/n}(r)$ provides the pressure gradients.

  If we assume ideal gas law and a constant ratio of P_{gas} & P_{rad}, we can also obtain the temperature profile $T(r)$.

- to solve (LE), we need to specify two boundary conditions

  (1) we want the central density to be finite: $\frac{d\rho}{dr} \to 0$ as $r \to 0$

  $\Rightarrow$ $\boxed{\frac{dD_n}{d\xi} = 0 \quad \text{at } \xi = 0}$

  (2) we want the density go to zero at the surface: $\rho(R) = 0$

  $\Rightarrow$ $\boxed{D_n(\xi_1) = 0 \quad \text{at surface } \xi = \xi_1}$

- we also require the normalisation
  $D_n(0) = 1$, so that ρ_c = central density

- total mass: $M = 4\pi \int_0^R \rho r^2 dr$

  radius of ★: $R = \lambda_n \xi_1$

  $\rightarrow M = 4\pi \int_0^{\xi_1} \lambda_n^2 \xi^2 \rho_c D_n^n(\xi) d(\lambda_n \xi)$

  $= 4\pi \lambda_n^3 \rho_c \int_0^{\xi_1} \xi^2 D_n^n d\xi$

  this integral can be solved directly.

  OR: we note that $\xi^2 D_n^n = -\frac{d}{d\xi}\left[\xi^2 \frac{dD_n}{d\xi}\right]$

  $\rightarrow M = -4\pi \lambda_n^3 \rho_c \xi_1^2 \left.\frac{dD_n}{d\xi}\right|_{\xi_1}$

  that means, the total mass can be derived by evaluating the derivative of D_n at the surface ξ_1.

- limitations: LE contains NO information about energy generation and energy transport in the star.

- There are three analytic solutions

  $n=0$: $\boxed{D_0(\xi) = 1 - \frac{\xi^2}{6} \quad \text{with } \xi_1 = \sqrt{6}}$

  $n=1$: $\boxed{D_1(\xi) = \frac{\sin\xi}{\xi} \quad \text{with } \xi_1 = \pi}$

  $n=5$: $\boxed{D_5(\xi) = \left(1 + \xi^2/3\right)^{-1/2} \quad \text{with } \xi_1 \to \infty}$

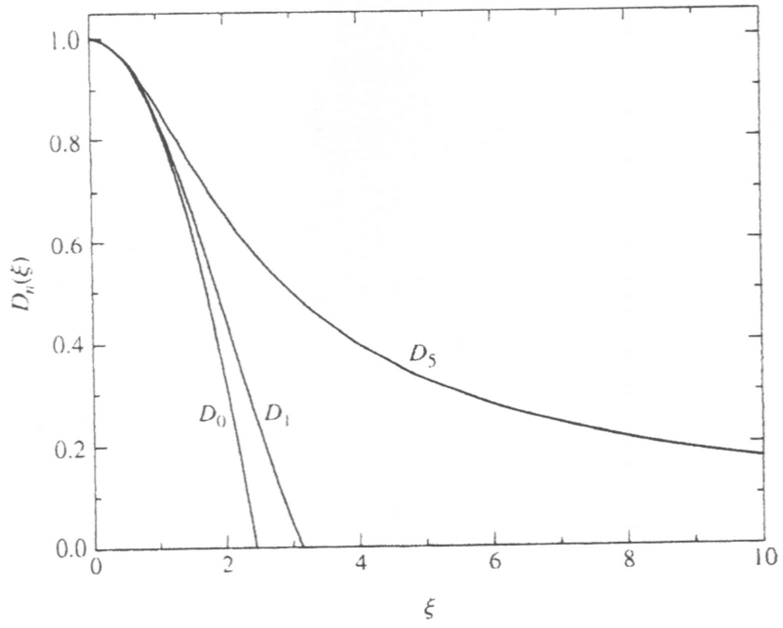

FIGURE 10.12 The analytic solutions to the Lane–Emden equation: $D_0(\xi)$, $D_1(\xi)$, and $D_5(\xi)$.

although $\xi_1 \to \infty$, the total mass remains finite for D_5. This is no longer the case for $n>5$!

↳ thus physically allowed: $\boxed{0 \leq n \leq 5}$

- the discussion of polytrops was originally motivated by adiabatic EOS. ($\gamma = \frac{5}{3} \,\hat{=}\, n = 1{,}5$)

  this holds very well for degenerat, non-relativistic objects, such as white dwarfs, here also (as discussed below): $\gamma = \frac{5}{3}$ & $n = \frac{3}{2}$

- $n = 3$ is the "Eddington" standard model of a star in radiative equilibrium.

  We have seen before that in this case

$$P = K \cdot \rho^{4/3}$$

Application of polytropes:

(A) White Dwarfs: - non-relativistic electron degeneracy

- $\gamma = 5/3 \Rightarrow n = \frac{3}{2}$

(B) Chandrasekhar limit for relativistic electron degeneracy

- $\gamma = 4/3 \Rightarrow n = 3$

$\hookrightarrow$ existence of critical mass!

The polytropic EOS with Lane-Emden equation leads to a relation between mass and radius:

$$\boxed{\left(\frac{GM}{M_n}\right)^{n-1} \left(\frac{R}{R_n}\right)^{3-n} = \text{CONSTANT}}$$ (**)

with constants $M_n = -\xi_1^2 \left.\frac{dD_n}{d\xi}\right|_{\xi_1} \cdot 4\pi \lambda_n^3 \rho_c$

$$R_n = \lambda_n \xi_1$$

Before going into how the stellar structure equations are solved in practice, we introduce a simplification which, albeit restrictive, turns out to be of both practical and pedagogical value.

7.2 Polytropic Equations of State and Polytropes

The primary, and classic, reference for the beginning portions of this section is Chandrasekhar (1939). Similar material, although not as exhaustive, may be found in Cox (1968, §23.1), and Kippenhahn and Weigert (1990, §19).

We shall first discuss polytropes in a general way but then interrupt the narrative to consider how these approximations to stellar models and, to some extent, real stars are calculated in practice. This last may seem to take us far afield but, toward the end of the section, we shall return to polytropes for a discussion of how they are used.

7.2.1 *General Properties of Polytropes*

In previous chapters we encountered equations of state where pressure was only a function of density (and, of course, composition). For example, the equation of state for a completely degenerate, nonrelativistic, electron gas was given by (3.61) as

$$P_e = 1.004 \times 10^{13} \left(\frac{\rho}{\mu_e}\right)^{5/3} \quad \text{dyne cm}^{-2} \tag{7.14}$$

which is a power law equation of state with $P \propto (\rho/\mu_e)^{5/3}$. We might then imagine a stellar model composed of a material for which μ_e is a constant throughout and in which *both* the equation of state *and* the actual run of pressure versus density satisfy (7.14). But if this condition is imposed beforehand it might conflict with the complete set of stellar structure equations and a self-consistent model would not be possible. *Polytropes* are pseudo-stellar models for which power law equations of pressure versus density such as (7.14) are assumed *a priori* but where no reference to heat transfer or thermal balance is made. Thus only the hydrostatic and mass equations are used and inconsistencies with respect to the complete set of stellar structure equations are avoided. This may seem to be a high price to pay for consistency, but the resulting polytropic structures have proven to be remarkably useful in the interpretation of many aspects of real stellar structure.

Another motivation for studying polytropes arises from consideration of the structure of certain types of adiabatic convection zones. In a region of efficient convection the actual "del" of (7.11) is given by $\nabla = \nabla_{ad} = 1 - 1/\Gamma_2$ (see eq. 3.88). If Γ_2 is assumed constant, then integrating (7.11) yields

$$P(r) \propto T^{\Gamma_2/(\Gamma_2-1)}(r). \tag{7.15}$$

If, in addition, the gas is an ideal gas with $T \propto P/\rho$, then $P(r) \propto \rho^{\Gamma_2}(r)$ and we have the same situation as above: P obeys a power law relation with respect to density as a function of radius.

In particular, we define a polytropic stellar model to be one in which the pressure is given by

$$P(r) = K\rho^{1+1/n}(r) \tag{7.16}$$

where n, the *polytropic index*, is a constant as is the proportionality constant K.[2] Since the polytrope is to be in hydrostatic equilibrium, then the distribution of pressure and density must be consistent with both the equation of hydrostatic equilibrium and conservation of mass. To best see how this works, divide the hydrostatic equation by ρ, multiply by r^2, and then take the derivative with respect to r of both sides to find

$$\frac{d}{dr}\left(\frac{r^2}{\rho}\frac{dP}{dr}\right) = -G\frac{d\mathcal{M}_r}{dr} = -4\pi G r^2 \rho$$

where the mass equation has been used to obtain the final equality. Rewrite this as

$$\frac{1}{r^2}\frac{d}{dr}\left(\frac{r^2}{\rho}\frac{dP}{dr}\right) = -4\pi G\rho \tag{7.17}$$

which is Poisson's equation. The latter identification is clear if we define the potential Φ such that $g = d\Phi/dr = G\mathcal{M}_r/r^2$, eliminate the pressure derivative (using 7.5), and find $\nabla^2\Phi = 4\pi G\rho$ in spherical coordinates.

We now perform a sequence of transformations with the intent of making (7.17) dimensionless. Define the dimensionless variable θ by

$$\rho(r) = \rho_c\,\theta^n(r) \tag{7.18}$$

where $\rho_c = \rho(r=0)$. The power law for pressure is then

$$P(r) = K\rho_c^{1+1/n}\theta^{n+1}(r) = P_c\,\theta^{1+n}(r). \tag{7.19}$$

The central pressure, P_c, is clearly equal to

$$P_c = K\rho_c^{1+1/n}. \tag{7.20}$$

Now substitute these into Poisson's equation and find the second-order differential equation for θ

$$\frac{(n+1)P_c}{4\pi G\rho_c^2}\,\frac{1}{r^2}\frac{d}{dr}\left(r^2\frac{d\theta}{dr}\right) = -\theta^n. \tag{7.21}$$

[2] Be careful not to confuse this n with the n used as the power law density exponent of opacity.

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(Hansen & Kawaler

Finally, introduce the new dimensionless radial coordinate, ξ, by

$$r = r_n \xi \tag{7.22}$$

where the scale length, r_n, is defined as

$$r_n^2 = \frac{(n+1)P_c}{4\pi G \rho_c^2}. \tag{7.23}$$

With this substitution, Poisson's equation becomes

$$\frac{1}{\xi^2}\frac{d}{d\xi}\left(\xi^2 \frac{d\theta}{d\xi}\right) = -\theta^n \tag{7.24}$$

and is now called the *Lane-Emden equation*. Models corresponding to solutions of this equation for a chosen n are called "polytropes of index n" and the solutions themselves are "Lane-Emden solutions" and are denoted by $\theta_n(\xi)$.[3]

Note that if the equation of state for the model material is an ideal gas with $P = \rho N_A kT/\mu$, then some easy manipulations yield

$$P(r) = K' T^{n+1}(r), \quad T(r) = T_c \theta(r) \tag{7.25}$$

with

$$K' = \left(\frac{N_A k}{\mu}\right)^{n+1} K^{-n}, \quad T_c = K \rho_c^{1/n} \left(\frac{N_A k}{\mu}\right)^{-1}. \tag{7.26}$$

Thus in a polytrope whose material equation of state is an ideal gas with constant μ, θ measures temperature. Finally, the radial scale factor in this case is

$$r_n^2 = \left(\frac{N_A k}{\mu}\right)^2 \frac{(n+1)T_c^2}{4\pi G P_c} = \frac{(n+1)K\rho_c^{1/n-1}}{4\pi G}. \tag{7.27}$$

To prepare complete polytropic models that might share some resemblance to stars, appropriate boundary conditions must be applied to the Lane-Emden equation. For a complete model, with center at $r=0$ and a surface that has vanishing density, these boundary conditions are as follows. For ρ_c in (7.18) to really be the central density, we require that $\theta(\xi=0)=1$. Furthermore, spherical symmetry at the center (dP/dr vanishing at $r=0$) requires that $\theta' \equiv d\theta/d\xi = 0$ at $\xi = 0$. This last condition pins down the solution at the center so that divergent solutions of the second-order system are suppressed. The regular solutions are called "E-solutions." If the surface is that place where $P=\rho=0$, then we require that the solution θ_n vanish there also. More specifically, the surface is where the *first zero* of θ_n occurs as measured from the center outward. (We do not want the pressure

[3] Note that we shall not always append the n subscript to θ_n. The index will usually be obvious from the context.

to vanish both at the "surface" and at some interior point.) We denote the location of the first zero by ξ_1 and it depends on the value of the polytropic index n. To summarize, the boundary conditions for a whole model are:

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$$\theta(0) = 1, \quad \theta'(0) = 0 \quad \text{at} \quad \xi = 0 \quad \text{(the center)} \tag{7.28}$$

$$\theta(\xi_1) = 0 \quad \text{at} \quad \xi = \xi_1 \quad \text{(the surface)}. \tag{7.29}$$

Since ξ_1 is the location of the surface, then the total (dimensional) radius is at

$$\mathcal{R} = r_n \xi_1 = \left[\frac{(n+1)P_c}{4\pi G \rho_c^2}\right]^{1/2} \xi_1. \tag{7.30}$$

Thus specifying K, n, and either ρ_c or P_c, yields the radius $\mathcal{R}$.

Analytic E-solutions for θ_n are obtainable for $n = 0$, 1, and 5. Numerical methods must be used to obtain solutions to the Lane-Emden equation for general n.

1. The solution for $n = 0$ is the constant-density sphere discussed in earlier chapters with $\rho(r) = \rho_c$. You may easily verify that

$$\theta_0(\xi) = 1 - \frac{\xi^2}{6}, \quad \text{with} \quad \xi_1 = \sqrt{6} \tag{7.31}$$

and $P(\xi) = P_c\theta(\xi) = P_c\left[1 - (\xi/\xi_1)^2\right]$. Except that we have not found P_c (which may be found once $\mathcal{M}$ and $\mathcal{R}$ are specified), this is the solution found for the constant-density sphere as given by (1.38). P_c is easily computed using (7.30) with $\xi_1 = \sqrt{6}$ to be $(3/8\pi)(G\mathcal{M}^2/\mathcal{R}^4)$ in accord with (1.39).

2. For $n = 1$, the solution θ_1 is the familiar "sinc" function

$$\theta_1(\xi) = \frac{\sin\xi}{\xi}, \quad \text{with} \quad \xi_1 = \pi. \tag{7.32}$$

The pressure and density follow from $\rho = \rho_c\theta$ and $P = P_c\theta^2$.

3. The polytrope for $n = 5$ has a finite central density but its radius is unbounded with

$$\theta_5(\xi) = \left[1 + \xi^2/3\right]^{-1/2} \quad \text{and} \quad \xi_1 \to \infty. \tag{7.33}$$

Despite the infinite radius, this polytrope does has a finite amount of mass associated with it.

(Nansen & Kawaler)

Complete and regular solutions with $n > 5$ are also infinite in extent but contain infinite mass. The range of n of interest to us for complete models is then $0 \le n \le 5$.

Given n and K, we can in principle find the dependence of P and ρ on ξ. However, we cannot obtain absolute physical numbers unless $\mathcal{R}$ and either ρ_c or P_c are first specified. This follows from (7.20) and (7.30). The main difficulty is that $\mathcal{R}$ is not known beforehand. But $\mathcal{M}$ is what we wish to specify and this turns out to be enough.

The mass contained in a sphere of radius r is found from (7.6) to be $\mathcal{M}_r = \int_0^r 4\pi r^2 \rho(r)\, dr$. In ξ-space this becomes

$$\mathcal{M}_\xi = 4\pi r_n^3 \rho_c \int_0^\xi \xi^2 \theta^n \, d\xi.$$

The integrand of this expression contains θ^n, but this is just the (negative) of the right-hand side of the Lane-Emden equation (7.24). Therefore insert the left-hand side of (7.24) in place of $-\theta^n$, notice that the factors of ξ^2 cancel and what is left is a perfect differential under the integral. The result is

$$\mathcal{M}_\xi = 4\pi r_n^3 \rho_c \left(-\xi^2\theta'\right)_\xi \tag{7.34}$$

where $\left(-\xi^2\theta'\right)_\xi$ means "evaluate $(-\xi^2\, d\theta/d\xi)$ at the point ξ." The total mass is given by $\mathcal{M} = \mathcal{M}(\xi_1)$. It should be clear that if $\mathcal{M}$ and $\mathcal{R}$ are specified in physical units, then all else follows. In what comes next, the relations between $\mathcal{M}$, $\mathcal{R}$, etc., are given without derivation. For example,

$$\mathcal{M} = (4\pi)^{-1/2} \left(\frac{n+1}{G}\right)^{3/2} \frac{P_c^{3/2}}{\rho_c^2} \left(-\xi^2\theta'\right)_{\xi_1} \tag{7.35}$$

which, in conjunction with (7.20), gives ρ_c or P_c in terms of $\mathcal{M}$. A little more algebra yields

$$\begin{aligned} P_c &= \frac{1}{4\pi(n+1)\left(\theta'\right)^2_{\xi_1}} \frac{G\mathcal{M}^2}{\mathcal{R}^4} \\ &= \frac{8.952 \times 10^{14}}{(n+1)\left(\theta'\right)^2_{\xi_1}} \left(\frac{\mathcal{M}}{\mathcal{M}_\odot}\right)^2 \left(\frac{\mathcal{R}}{\mathcal{R}_\odot}\right)^{-4} \quad \text{dyne cm}^{-2}. \end{aligned} \tag{7.36}$$

Note that the last result requires n, but not K.

Another result that will prove useful follows from solving for K given n, $\mathcal{M}$, and $\mathcal{R}$:

$$K = \left[\frac{4\pi}{\xi^{n+1}\left(-\theta'\right)^{n-1}}\right]^{1/n}_{\xi_1} \frac{G}{n+1} \mathcal{M}^{1-1/n} \mathcal{R}^{-1+3/n}. \tag{7.37}$$

Note that if $n = 3$, K depends only on $\mathcal{M}$ or, turned around, $\mathcal{M}$ does not depend on $\mathcal{R}$ for any K if $n = 3$.

2gc

If the equation of state is that of an ideal gas, then the central temperature is given by

$$\begin{aligned} T_c &= \frac{1}{(n+1)\left(-\xi\theta'\right)_{\xi_1}} \frac{G\mu}{N_A k} \frac{\mathcal{M}}{\mathcal{R}} \\ &= \frac{2.293 \times 10^7}{(n+1)\left(-\xi\theta'\right)_{\xi_1}} \mu \left(\frac{\mathcal{M}}{\mathcal{M}_\odot}\right) \left(\frac{\mathcal{R}}{\mathcal{R}_\odot}\right)^{-1} \quad \text{K}. \end{aligned} \tag{7.38}$$

You may easily verify that T_c for the constant-density sphere ($n = 0$) is the same as given by the earlier result (1.53, and see Table 7.1).

A useful quantity that depends only on n is the ratio of central density to mean density. This is given by

$$\frac{\rho_c}{\langle\rho\rangle} = \frac{1}{3} \left(\frac{\xi}{-\theta'}\right)_{\xi_1}. \tag{7.39}$$

Thus the statement will sometimes be made that "this stellar model looks like a polytrope of index so-and-so because its degree of central concentration is such-and-such"; that is, comparison of central to mean density implies an n by way of (7.39). This is often a useful way to look at things—if you know what you're doing.

Finally, it is an easy matter to show that the gravitational potential energy of a polytrope is (see Eq. 1.6 and the discussion preceding that equation for a refresher on Ω)

$$\Omega = -\frac{3}{5-n} \frac{G\mathcal{M}^2}{\mathcal{R}}. \tag{7.40}$$

For the constant-density sphere the coefficient $3/(5-n)$ is just 3/5 and this is the value quoted for the quantity "q" after (1.7).

Now that some of the formalism is out of the way, what are interesting values for n? The pressure of the completely degenerate but nonrelativistic electron gas goes as $\rho^{5/3}$. Hence, by the definition of the polytropic equation of state (7.16), n for this case is 1.5 (or "a three-halves polytrope"). The density exponent for the fully relativistic case is 4/3 and thus $n = 3$ (or "an n equal three polytrope"). The same indices crop up in other applications as we shall see. For now, recall that $P \propto \rho^{\Gamma_2}$ in an ideal gas convection zone. If no ionization is taking place (almost a contradiction for a real convection zone) then $\Gamma_2 = 5/3$ and $n = 3/2$ again. It will turn out that indices of 1.5 and 3 are the ones usually encountered in simple situations. How unfortunate it then is that neither of these values have analytic E-functions associated with them. Therefore, how *are* these nonanalytic cases computed? The following subsection looks into this question and serves as a brief introduction to how stellar models are computed. After this, we shall use the results from the polytropic calculations.

(Hansen & Kawaler)

for $n=3$ (relativistic degeneracy or radiation dominated star)

the dependence on radius r vanishes!

↳ there is only one possible value for the mass given in terms of K:

$$\boxed{M_{crit} = 4\pi M_3 \left(\frac{K}{\pi G}\right)^{3/2}}$$

only this mass satisfies hydrostatic equilibrium

↳ substitution of constants & parameters:

$$M_{crit} = \frac{5{,}83}{\mu_e^2} M_\odot$$

↳ for hydrogen-poor stars we get $\mu_e = 2$

and

$$\boxed{M_{crit} = 1{,}46 \ M_\odot}$$

Chandrasekhar mass

NB: this limiting mass can also be derived from the virial theorem:

- recall the condition for hydrostatic equilibrium:

$$\frac{dP}{dr} = -\rho(r)\,\frac{GM(r)}{r^2}$$

$\bar{\rho} = \mu m_p \cdot n$

$$\Rightarrow \quad \frac{P}{R} \approx \bar{\rho}\,\frac{GM}{R^2} \longrightarrow P \approx \rho\,\frac{GM}{R} = n \cdot \frac{GMm_p}{R}\cdot\mu$$

- Fermi energy:

$n \approx \frac{1}{R^3}\frac{M}{\mu m_p}$

$$p_F = \hbar k_F = \hbar n^{1/3} = \hbar\left(\frac{M}{\mu m_p R^3}\right)^{1/3} \quad \text{(Fermi momentum)}$$

$$E_F = n p_F c = \hbar c\left(\frac{M}{\mu m_p}\right)^{1/3}\frac{1}{R}\cdot n \quad \text{(Fermi energy)}$$

$\Rightarrow$ in balance:

$$P \approx \frac{GM\mu m_p}{\cancel{R}}\cdot\cancel{n} \approx \hbar c\left(\frac{M}{\mu m_p}\right)^{1/3}\frac{1}{\cancel{R}}\cdot\cancel{n}$$

$$\Rightarrow \quad G^3M^3(\mu m_p)^3 \approx (\hbar c)^3\,\frac{M}{\mu m_p}$$

$$\Rightarrow \quad \frac{M^2}{m_p^2} \approx \frac{(\hbar c)^3}{G^3 m_p^6} \qquad \text{with } \mu \approx 1$$

$$\Rightarrow \quad \boxed{M \approx m_p \cdot \frac{1}{\alpha_G^{3/2}}} \qquad \alpha_G = \frac{Gm_p}{\hbar c} = 6\cdot 10^{-39}$$

similar to the radiation dominated case, there is a critical mass for relativistic electron degeneracy!! $\Rightarrow$ Chandrasekhar mass now consider $n = 1{,}5$ polytropes: $\boxed{\text{non-relativistic degenerate EOS}}$

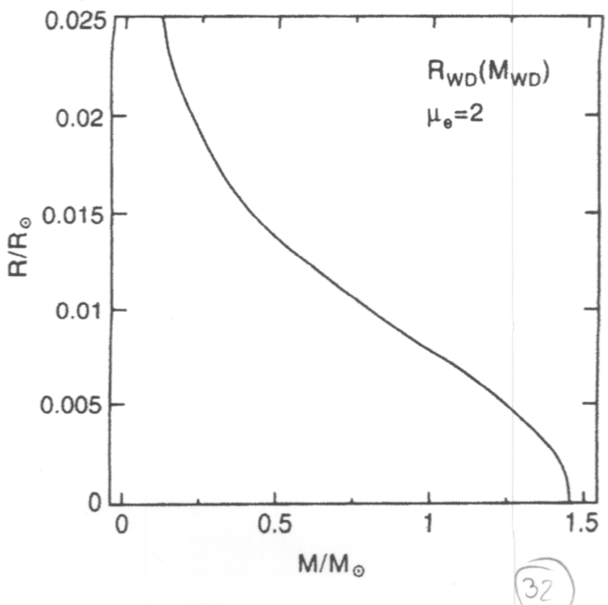
- examples: White dwarfs
- from (**) we get $R \propto M^{-1/3}$

$$\bar{\rho} \propto M R^{-3} \propto M^2$$

- sequence: let's start with a non-rel. degenerate (low-mass) white dwarf

* if we add mass, material in the center gets denser, because the radius shrinks.
* n will vary throughout the star; in the interior, you may reach $n = 3$
* as you add up more and more mass, more of the white dwarf will be relativistic
* eventually everything is $n = 3$; But: stability only for one mass: $M \approx 1{,}46\, M_\odot$
* if the mass is larger WD collapse sets in!

Mass-Radius relationship for white dwarfs



7. Energy Transport

There are <u>three</u> mechanisms to transport energy in the stellar interior:

- <u>RADIATION:</u> Energy transport by photons

  [- photons can be absorbed and re-emitted in different wavelength and direction

  - coupling to matter via <u>opacity κ.</u>]

- <u>CONVECTION:</u> energy is transported via bulk motion in the fluid

  [- buoyant mass element carry excess heat outwards, while cooler elements move towards the center]

- <u>CONDUCTION:</u> heat transport via collisions between gas particles

  [- usually is not important in stars]

Radiative Temperature Gradient 1

- radiation pressure gradient:

$$\boxed{\frac{dP_{rad}}{dr} = -\frac{\bar{\varkappa}\rho}{c} F_{rad}}$$

where F_{rad} = outwards radiative flux, ρ = density, c = speed of light, and the material constant $\varkappa$ = opacity.

- the opacity $\varkappa$ describes coupling between radiation and matter

- recall: for blackbody radiation, the radiation pressure is 1/3 of the energy density: $\boxed{P_{rad} = \frac{1}{3} u}$

- opacity = absorption coefficient:

  ↳ in radiative transfer equation $\boxed{dI_\nu = -\varkappa_\nu \rho I_\nu ds}$, the intensity at frequency ν is attenuated along a distance ds by the factor $\varkappa_\nu \rho$.

↳ recall: formal integration gives

$$\boxed{I_\nu = I_{\nu,0} \cdot e^{-\varkappa \varrho s}} \quad (*)$$

↳ recall also: for scattered photons, the characteristic distance ℓ is the mean free path of photons:

$$\boxed{\ell = \frac{1}{\varkappa_\nu \varrho} = \frac{1}{n \sigma_\nu}}$$

with σ_ν = interaction cross section (e.g. via Thompson scattering

↳ definition of optical depth:

$$d\tau_\nu = -\varkappa_\nu \varrho \, ds$$

along a light ray

↳ formal integration turns into

$$\boxed{I_\nu(s) = I_\nu(0)\, e^{-\varkappa \varrho s}}$$

$$\hat{=} \quad \boxed{I_\nu(\tau) = I_\nu(0)\, e^{-\tau}}$$