optical depth = number of mean free paths through the medium.

$\tau_\nu \ll 1$: OPTICALLY thin LIMIT

(light with frequency ν passes through medium essentially unattenuated.)

$\tau \gg$: OPTICALLY thick LIMIT

(light gets "caught" in the medium, you can't see through.)

- Sources of opacity:

1. BOUND-BOUND-TRANSITIONS (excitation & de-excitation)
2. BOUND-FREE ABSORPTION (photoionization)
3. FREE-FREE ABSORPTION (scattering)
4. Electron SCATTERING

[scattering by free electron: Thomson

scattering by loosely-bound electron:

Compton if $\lambda \ll R_{atom}$

Rayleigh if $\lambda \gg R_{photon}$]

Rosseland Mean Intensity

- It is often useful to employ an opacity that is averaged over all wavelength (or frequencies) to produce a function that depends only on composition, density, and temperature.

- the most common approach is to compute a harmonic mean weighted with the Planck function:

$$\boxed{\frac{1}{\bar{\kappa}} = \frac{\int_0^\infty \frac{1}{\kappa_\nu} \frac{\partial B_\nu(T)}{\partial T} d\nu}{\int_0^\infty \frac{\partial B_\nu(T)}{\partial T} d\nu}}$$

Rosseland mean opacity

- there is no analytic solution, however, there are analytic approximations (fits) to the various processe that may contribute to $\bar{\kappa}$:

↳ bound-free opacity:

$$\boxed{\bar{\kappa}_{bf} = 4,34 \times 10^{22} \frac{g_{bf}}{t} Z(1+X) \frac{\rho}{T^{3,5}} \, cm^2 g^{-1}}$$

↳ free-free opacity:

$$\boxed{\bar{\kappa}_{ff} = 3{,}68 \cdot 10^{19} \, g_{ff} \, (1-Z)(1+X) \, \frac{\rho}{T^{3,5}} \quad cm^2 g^{-1}}$$

↳ electron scattering

$$\boxed{\bar{\kappa}_{es} = 0{,}2 \, (1+X) \quad cm^2 \, g^{-1}}$$

↳ H^-:

$$\boxed{\bar{\kappa}_{H^-} \approx 7{,}9 \cdot 10^{-33} \left(\frac{Z}{0{,}02}\right) \rho^{1/2} T^{9} \quad cm^2 g^{-1}}$$

- total Rosseland mean:

$$\bar{\kappa} = \overline{\kappa_{bb} + \kappa_{bf} + \kappa_{ff} + \kappa_{es} + \kappa_{H^-}}$$

$$\quad (A) \quad (B) \quad (C) \quad (D) \quad (E)$$

(A) there is no analytic fit to (A) because of complexity of contributions from different lines. → best from Tables.

(B) + (C) have the well known form of

Kramer's opacity law:

$$\boxed{\bar{\kappa} = \kappa_0 \cdot \rho / T^{3,5}}$$

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X, Y, Z = mass fractions of H, He, and the rest $(X+Y+Z=1)$

Gaunt factors g_{bf} & g_{ff} are quantum-mechanical corrections: $g \approx 1$

t = guillotine factor, describes decrease of opacity after atoms have been ionized. $(t = 1 \ldots 100)$

(D) Because electron scattering is wavelength independent κ_{es} has a particularly simple form.

(E) in cooler stars such as the Sun bf and ff transitions of H^- contribute significantly to the opacity

in the range $3000\,K \leq T \leq 6000\,K$

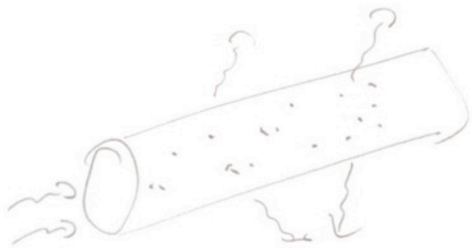
&

$10^{-10}\, g/cm^3 \leq \rho \leq 10^{-5}\, g/cm^3$

κ_{H^-} increases with T as T^9!

NB:

(D) Electron scattering:



$dI_\nu = -\varkappa_\nu \rho I_\nu \, ds$

simple geometry:

$\varkappa = \frac{\sigma_e n_e}{\rho} \quad [cm^2 g^{-1}]$

the cross section is the classical electron radius: $\sigma_e = \frac{8\pi}{3}\left(\frac{e^2}{mc^2}\right)^2$

$= 0{,}6652 \cdot 10^{-24} \, cm^2$

↳ as long as photon energy is below $m_e c^2$, the opacity of Tomson scattering is independent of ν.

↳ number of e^- comes from ionisation of hydrogen mostly

(C) - electron can absorb photon only when ion is present

- for computing $\varkappa_{ff}$ turn problem around and study bremsstrahlung

↓

$j_\rho \propto n_e n_i$

(40)

- integrate over all electron (velocity) states
  ↳ integrate Maxwell-Boltzmann distribution fct
    ↳ $4\pi j_\omega \varsigma d\omega = \frac{2\pi}{3} \frac{Z_c^2 e^6}{m_e c^2} \left(\frac{2\pi}{m_e kT}\right)^{1/2} n_e n_i e^{-\hbar\omega/kT} d\omega$

  ↳ yields $j \propto T^{-1/2}$ $\overset{!}{\triangledown}$

- together: $j\varsigma \propto n_e n_i T^{+1/2}$ (note change in sign!) → fully ionised H: $n_e = n_i = \frac{\varsigma}{m_p}$
$= n$

- in equilibrium: $dI = (-\bar{\varkappa}\varsigma I + j\varsigma) ds \overset{!}{=} 0$
  ↳ $\boxed{\bar{\varkappa}_{ff} = \frac{j}{I}}$

- in LTE: $I = S = B(T)$ Planck Function (integrated)
  $= a \cdot T^4$ (Stefan-Boltzmann law)
  ↳ $\boxed{\bar{\varkappa}_{ff} \propto \frac{nT^{1/2}}{T^4} \propto n T^{-3,5}}$ Kramers opacity law

(B) similar for bf: $\varkappa_{bf} \propto n T^{-3,5}$ $\overset{!}{\triangledown}$

Radiative Temperature Gradient 2:

- now that we have studied possible sources of opacity in the stellar interior, we can turn back to computing the temperature stratification in the star.

- radiation pressure gradient:

$$\frac{dP_{rad}}{dr} = -\frac{\bar{\kappa}\rho}{c} F_{rad}$$

- from $P_{rad} = \frac{1}{3} u_{rad} = \frac{a}{3} T^4$ we also get

$$\frac{dP_{rad}}{dr} = \frac{4}{3} a T^3 \frac{dT}{dr}$$

- combining both we get for $\frac{dT}{dr}$:

$$\frac{dT}{dr} = -\frac{3}{4ac} \frac{\bar{\kappa}\rho}{T^3} F_{rad}$$

- we can now express the radiative flux in terms of the local radiative luminosity at radius r:

$$F_{rad} = \frac{L_r}{4\pi r^2}$$

- thus: $$\boxed{\frac{dT}{dr} = -\frac{3}{4ac} \frac{\bar{\kappa}\rho}{T^3} \frac{L_r}{4\pi r^2}}$$

temperature gradient for radiative transport.

Example: A simple mass-luminosity relation based on Thomson scattering (for hot, high-mass $\star$'s)

$\hookrightarrow$ $L \propto M^3 \mu^5$

- $\star$ is hot in center: $kT_c \approx \frac{\mu G M m_p}{R}$ $\boxed{\square}$
cold at surface $\hookrightarrow$ heat transport

- energy flux $F_{rad} = -\frac{c}{\bar{\kappa}\rho} \cdot \frac{dP_{rad}}{dr}$

$$= -\frac{c}{\bar{\kappa}\rho} \cdot \frac{4}{3} a T^3 \frac{dT}{dr}$$

- Electron scattering (Thomson):

$$\bar{\kappa} = \frac{\sigma_e n_e}{\rho} \qquad \text{with } \sigma_e = \frac{8\pi}{3}\left(\frac{e^2}{mc^2}\right)^2$$

$\hookrightarrow$ $F_{rad} = -\frac{c}{n_e \sigma_e} \cdot \frac{4}{3} a T^3 \frac{dT}{dr}$

- with $\boxed{\square}$:

$$F_{rad} \approx -\frac{4}{3}\frac{ca}{n_e\sigma_e} \cdot T_c^3 \frac{T_c}{R}$$

$$\approx -\frac{4}{3}\frac{ca}{n_e\sigma_e} \cdot \left(\frac{\mu G M m_p}{kR}\right)^4 \cdot \frac{1}{R}$$

- luminosity: $L \approx 4\pi R^2 \cdot F$

$$\approx \frac{16\pi}{3} R^2 \cdot \frac{ac}{\sigma_e} \cdot \frac{\mu m_p}{\rho} \cdot \left(\frac{\mu G M m_p}{kR}\right)^4 \frac{1}{R}$$

drop all constants and use $\rho \propto M/R^3$

$$L \propto R^2 \cdot \frac{R^3}{M} \frac{M^4}{R^5} \cdot \mu^5$$

$\hookrightarrow$ $\boxed{L \propto M^3 \mu^5}$

- Plugging in all constants, gives a prefactor not that far from $L_\odot = 4 \cdot 10^{33}$ erg/s for $M_\odot = 2 \cdot 10^{33}$ g.

- Dependencies:

– When Thomson scattering dominates opacity ($M > M_\odot$) L is independent of stellar radius and proportional to M^3.

– Composition determins luminosity, also:

* pure H: $P = \frac{2kT m_p}{\rho} \rightarrow \mu = 1/2$

* pure He: $P = n_\alpha kT + 2 n_\alpha kT = 3 n_\alpha kT$

$$= \frac{3}{4} \frac{\rho kT}{m_p} \rightarrow \mu = 4/3$$

(L↑ for He)

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Example: Mass luminosity relation for low-mass $*$'s (where Kramer's opacity dominates)

- Again: $F_{rad} = -\frac{c}{\varkappa\rho}\frac{4}{3}aT^3\frac{dT}{dr}$
- Use $\varkappa \propto \rho T^{-3,5}$

$$\Rightarrow F_{rad} \propto -\frac{4}{3}ca\underbrace{\rho^{-2}}_{\approx\left(\frac{M}{R^3}\right)^2}\underbrace{T^{6,5}}_{T_c^{6,5}}\underbrace{\frac{dT}{dr}}_{\approx\frac{T_c}{R}}$$

$$\propto \frac{R^6}{M^2}\cdot\frac{T_c^{7,5}}{R} \qquad \left(T_c \approx \frac{GM\mu m_p}{kR}\right)$$

$$\propto \frac{R^6}{M^2}\frac{M^{7,5}}{R^{8,5}} \propto \frac{M^{5,5}}{R^{2,5}}$$

- the luminosity: $L \approx 4\pi R^2 \cdot F_{rad}$

$$\Rightarrow L \propto R^2\cdot\frac{M^{5,5}}{R^{2,5}} \longrightarrow \boxed{L \propto \frac{M^{5,5}}{R^{0,5}}}$$

- Low-mass stars have very strong mass dependence $\Rightarrow L \propto M^{5,5}$ (R = const.)

& luminosity also depends on stellar radius $\Rightarrow L \propto R^{-0,5}$ (for M = const.)

Table of ZAMS parameters
[from Shore: "Tapestry of Mod. Astrophys"]

$\frac{M}{M_\odot}$	$\log \frac{L}{L_\odot}$	$\log \frac{T_{eff}}{K}$	$\frac{T_c}{10^7 K}$	$\frac{R}{R_\odot}$
0,3	− 1,90	3,55	0,8	0,3
0,5	− 1,42	3,59	0,9	0,44
1,0	− 0,15 *	3,75	1,4	0,9 *
2,0	1,22	3,97	2,2	1,6
3,0	1,91	4,09	2,4	2,0
5,0	2,74	4,24	2,8	2,6
9,0	3,61	4,38	3,1	3,6
15,0	4,32	4,51	3,4	4,7

* our Sun is already chemically evolved.

Pressure Scale Height

- before we turn to convective transport, it is useful to define a characteristic lengthscale associated with pressure gradients: Pressure Scale Height

$$\boxed{\frac{1}{H_p} = -\frac{1}{P}\frac{dP}{dr}}$$

the ⊖ accounts for the fact that $P(r)$ decreases outwards

- if we assume H_p = const. throughout star we get (in linear approximation)

$$\boxed{P = P_0 \cdot e^{-r/H_p}}$$

↳ H_p = distance over which gas pressure decreases by factor of e.

- hydrostatic equilibrium (pressure grad. balances gravity): $\frac{dP}{dr} = -g\rho = -\frac{GM(r)}{r^2}\cdot\rho$

↳ $\boxed{H_p = \frac{P}{\rho g} = \frac{P r^2}{G M(r) \rho}}$

Note: a typical value for the Sun is

$$H_p \approx \frac{1}{10} R_\odot \; !$$

Adiabatic Temperature Gradient

- Start with ideal gas law: $(PV = NkT)$

$$P_{gas} = \frac{\rho k T}{\mu m_p}$$

μ = mean mol. weight

$V = \frac{1}{\rho}$ = specific volume

$$\hookrightarrow \boxed{\frac{dP_{gas}}{dr} = -\frac{P}{\mu}\frac{d\mu}{dr} + \frac{P}{\rho}\frac{d\rho}{dr} + \frac{P}{T}\frac{dT}{dr}}$$

- use polytropic EOS: $P = K\rho^{\gamma}$

$$\hookrightarrow \boxed{\frac{dP}{dr} = \gamma \frac{P}{\rho}\frac{d\rho}{dr}}$$

- let's assume now $\mu = \text{const}$, then we can combine both equations:

$$\gamma \frac{P}{\rho}\frac{d\rho}{dr} = \frac{P}{\rho}\frac{d\rho}{dr} + \frac{P}{T}\frac{dT}{dr}$$

$$\hookrightarrow \boxed{\left.\frac{dT}{dr}\right|_{ad} = (\gamma - 1)\frac{T}{\rho}\frac{d\rho}{dr}}$$

expressed in terms of $d\rho/dr$

OR: $$\boxed{\left.\frac{dT}{dr}\right|_{ad} = \left(1 - \frac{1}{\gamma}\right)\frac{T}{P}\frac{dP}{dr}}$$ ★★★

in hydrostatic balance, we can express $\frac{dP}{dr}$ by the gravitational force: $\frac{dP}{dr} = -\frac{GM}{r^2}\rho$

$$\hookrightarrow \quad \left.\frac{dT}{dr}\right|_{ad} = -\left(1-\frac{1}{\gamma}\right)\frac{T}{P}\frac{GM}{r^2}\rho$$

with ideal gas law: $P = \frac{\rho k T}{m_p \mu}$ we finally get

$$\hookrightarrow \quad \boxed{\left.\frac{dT}{dr}\right|_{ad} = -\left(1-\frac{1}{\gamma}\right)\frac{\mu m_p}{k}\frac{GM(r)}{r^2}}$$

adiabatic temperature gradient

- rephrase using: $g = \frac{GM(r)}{r^2}$ and $\frac{k}{\mu m_p} = nR$ and $\gamma = \frac{c_p}{c_v}$ and $c_p - c_v = nR$

$$\hookrightarrow \quad \boxed{\left.\frac{dT}{dr}\right|_{ad} = -\frac{g}{c_p}}$$

- think of a hot bubble that rises due to buoyancy effects: if it rises fast enough such that heat transfer with surrounding is negligible, it evolves adiabatically.

it is in pressure equilibrium with its surrounding (if sound waves have sufficient time to travel across, which is usually the case).

after some time, it finally thermalises with surrounding and loses its identity and dissolves.

- the temperature evolution of bubble is given by $\boxed{\left.\frac{dT}{dr}\right|_{ad} = -\frac{g}{c_p}}$.

- if the actual temperature gradient in the star is LARGER than the adiabatic value, i.e. if it is SUPERADIABATIC, then CONVECTION sets in.

[buoyancy effects grow as the bubble climbs up and the upwards motion continues
↳ convective instability]

- Criterium for convection:

$$\boxed{\left|\frac{dT}{dr}\right|_{act} > \left|\frac{dT}{dr}\right|_{ad}}$$

convection sets in even if $\left|\frac{dT}{dr}\right|_{act}$ is only slightly larger than $\left|\frac{dT}{dr}\right|_{ad}$. this is sufficient to carry almost all energy outwards by convection

- if convection sets in, it brings the effective temperature gradient close to the convective / adiabatic value

$$\hookrightarrow \boxed{\left|\frac{dT}{dr}\right|_{eff} \approx \left|\frac{dT}{dr}\right|_{ad}}$$

- fully convective stars can be well described as polytropes with $\gamma = 5/3$, i.e. with index $n = \frac{1}{\gamma - 1} = 3/2$.

- we can find another criterion for the onset of convection:

- Revisit equation (***): $\frac{dT}{dr}\Big|_{ad} = \left(1 - \frac{1}{\gamma}\right)\frac{T}{P}\frac{dP}{dr}$

- and recall: $\left|\frac{dT}{dr}\right|_{act} > \left|\frac{dT}{dr}\right|_{ad}$ for instability

$$\hookrightarrow \text{combine: } \left|\underbrace{\frac{dT}{dr}\Big|_{act}}_{<0}\right| > \left|\underbrace{\left(1 - \frac{1}{\gamma}\right)}_{>0}\frac{T}{P}\underbrace{\frac{dP}{dr}}_{<0}\right|$$

- both T and P decrease with increasing radius r

- $\left(1-\frac{1}{\gamma}\right)>0$ as $\gamma<1$

$$\Rightarrow \quad \frac{dT}{\cancel{dr}}\bigg|_{act} > \left(1-\frac{1}{\gamma}\right)\frac{T}{P}\frac{dP}{\cancel{dr}}$$

$$\frac{P\,dT}{T\,dP} > 1-\frac{1}{\gamma}$$

$$\Rightarrow \quad \frac{d\ln T}{d\ln P} > \frac{\gamma-1}{\gamma} \longrightarrow \boxed{\frac{d\ln P}{d\ln T} < \frac{\gamma}{\gamma-1}}$$

- for an ideal monoatomic gas, convection will occur in regions of the star where

$$\boxed{\frac{d\ln P}{d\ln T} < 2{,}5}$$

(recall $\gamma = 5/3$)

in this case the temperature gradient is

$$\boxed{\frac{dT}{dr} \simeq \frac{dT}{dr}\bigg|_{ad} = -\left(1-\frac{1}{\gamma}\right)\frac{\mu m_p}{k}\frac{GM_r}{r}} \quad \text{(A)}$$

Otherwise, radiation dominates and

$$\boxed{\frac{dT}{dr} = -\frac{3}{4ac}\frac{\bar{\kappa}\rho}{T^3}\frac{L_r}{4\pi r^2}} \quad \text{(B)}$$

• another way of looking at the question of when convection dominates over radiation comes from comparing (A) with (B):

$$-\left(1-\frac{1}{\gamma}\right)\frac{\mu m_p}{k}\frac{GM_r}{r} \approx -\frac{3}{4ac}\frac{\bar{\kappa}\rho}{T^3}\frac{L_r}{4\pi r^2}$$

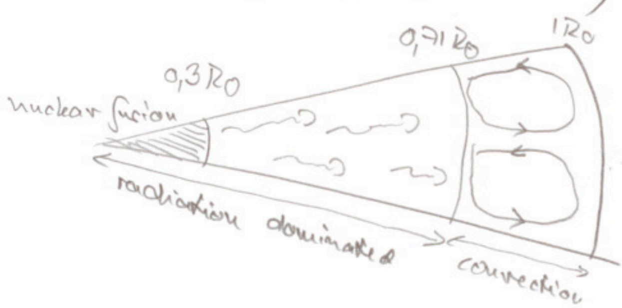
Convection occurs:

(1) when the stellar opacity is large, then an unachievable temperature gradient would be necessary for transporting away all energy by radiation
↳ convection then is more efficient for energy transport.

(2) in regions where ionization occurs, because of large specific heat and low adiabatic gradients.

(3) when the temperature dependence of nuclear energy generation is large, causing steep radiative flux gradients and thus a large temperature difference.

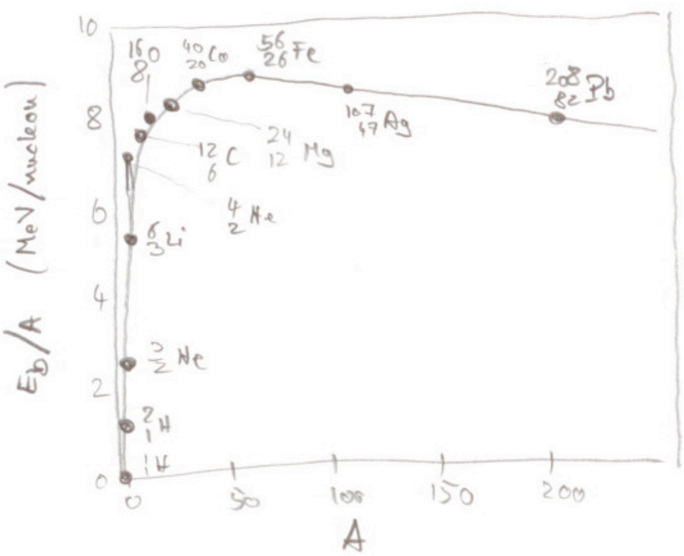
In stellar atmospheres often (1) + (2) are met simultaneously (e.g. Sun)



8. Nuclear Energy Source

- For nuclei less massive than iron, nuclear fusion releases energy.

  For heavier elements, energy is released by "fission"



- Example: Can nuclear fusion power Sun for long enough?

Assume: - initially 100% hydrogen
- only inner 10% of Sun can burn H to He
- 0,7% of H mass gets converted into energy when forming He.

$\hookrightarrow E_{nucl} \approx 0{,}1 \cdot 0{,}007 \cdot 1 M_\odot c^2$

$\approx 1{,}3 \cdot 10^{51}$ erg

$\hookrightarrow$ time scale: $t_{nucl} \approx \frac{E_{nucl}}{L_\odot} \approx \frac{1{,}3 \cdot 10^{51} \text{ erg}}{4 \cdot 10^{33} \text{ erg}} s$

$\approx 3{,}25 \cdot 10^{17}$ s

$\approx 10^{10}$ yr

compare to grav. contraction timescale:

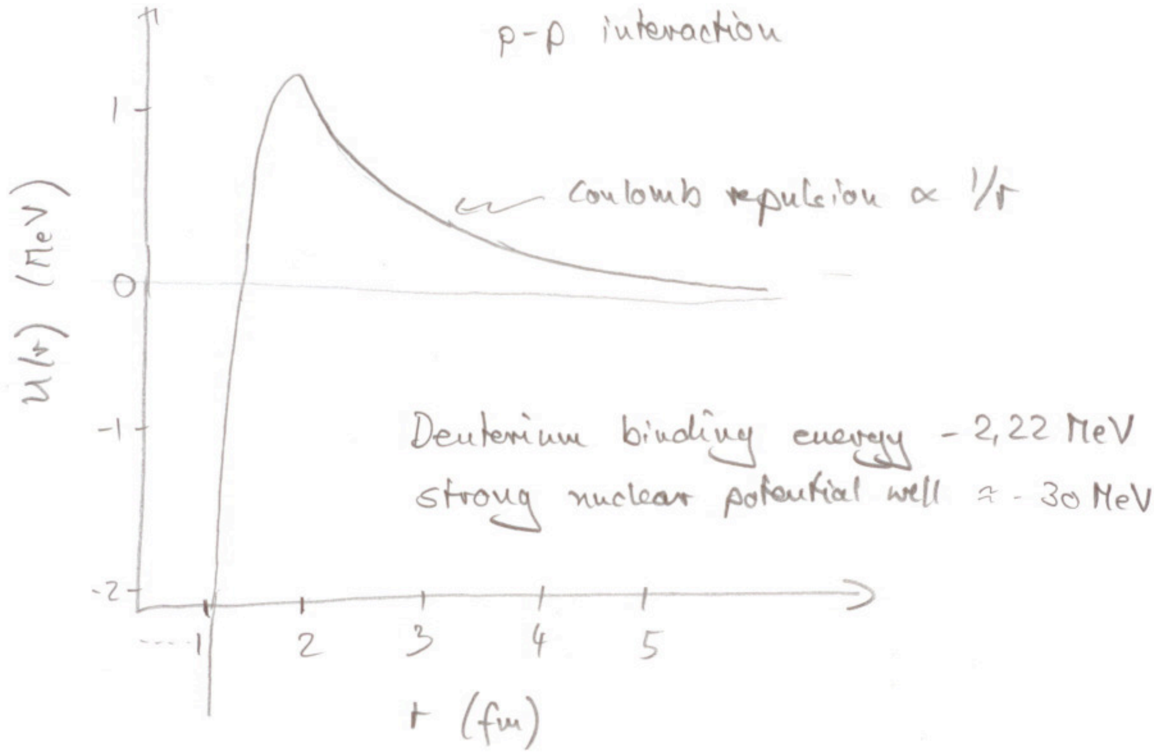
$t_{\text{Kelvin-Helmholtz}} \approx 30 \cdot 10^6$ yr

$\hookrightarrow$ $t_{nucl} \gg t_{KH}$

- Quantum mechanical tunneling is necessary for nuclear fusion to work in the stellar interior.

to show that, check temperature required in the fully classical approximation.

we ask: what temperature is required for two protons to come close enough so that strong forces overwhelm Coulomb repulsion?



use Maxwell-Boltzmann velocity distribution.

↳ peak: $\frac{1}{2}\mu_m \overline{v^2} = \frac{3}{2}kT = \frac{1}{4\pi\epsilon_0}\frac{Z_1 Z_2 e^2}{r}$ $\qquad \mu_m$ = reduced mass

↳ $\boxed{T = \frac{Z_1 Z_2 e^2}{6\pi\epsilon_0 k r}}$

for $Z_1 = Z_2 = 1$ and $r \approx 1\,fm = 10^{-15}\,m$

↳ $T \approx 10^{10}\,K$

this is too hot compared to $T_c = 14 \cdot 10^6\,K$ of the Sun, even if Maxwell-tail is considered (instead of peak)

$\longrightarrow$ other process required for fusion

$\hookrightarrow$ <u>tunneling</u>

<u>Approximation:</u> tunneling becomes important when De Broglie wavelength of particle gets of order of the extent of potential well, i.e. of order fm.

$\hookrightarrow$ kinetic energy: $\frac{1}{2}\mu_m v^2 = \frac{p^2}{2\mu_m} = \frac{(h/\lambda)^2}{2\mu_m}$

[Heisenberg: $\lambda \cdot p_\lambda \approx h$]

$\hookrightarrow$ put together: $\frac{1}{4\pi\epsilon_0}\frac{Z_1 Z_2 e^2}{\lambda} = \frac{(h/\lambda)^2}{2\mu_m}$

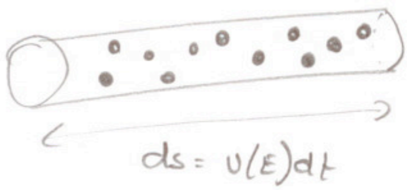
$\hookrightarrow$ $\boxed{T = \frac{Z_1^2 Z_2^2 e^4 \mu_m}{12\pi\epsilon_0^2 h^2 k}}$

Again for collision of two protons: $\mu_m = \frac{1}{2} m_p$ and $Z_1 = Z_2 = 1$

$\hookrightarrow$ $T \approx 10^7$ K

this temperature is consistent with estimates of stellar interior

- Nuclear reaction rate & Gamov peak



number of reactions per nucleus per time intervall

= probability for reaction × incident flux

↳ $dN_E = \sigma(E)\, v(E)\, n_i\, dE\, dt$

$\sigma(E)$ = cross section
$v(E)$ = velocity, corresponding to E
n_i = # of incident particles with energy E

↳ $v(E) = \sqrt{\frac{2E}{\mu m}}$

↳ number of reactions per second per nucleus

$$\frac{dN_E}{dt} = \sigma(E)\, v(E)\, \frac{n_i}{n}\, n_E\, dE$$

$$n_E\, dE = \frac{2n}{\sqrt{\pi}} \frac{1}{\sqrt{kT}^3} E^{1/2} e^{-E/kT} dE$$

= # of particles with E in $[E, E+dE]$

= from Maxwell-Boltzmann distribution

↳ reaction rate between incident flux $n_i v(E)$ and n_x targets is then

$$R_{ix} = \int_0^\infty n_x n_i \sigma(E) v(E) \frac{n_E}{n} dE$$

↳ now obtain $\sigma(E)$: with tunneling probability, we get

$$\sigma(E) \propto e^{-2\pi U_c/E} \quad \text{(A)}$$

where U_c = height of Coulomb barrier.

because $\frac{U_c}{E} = \frac{Z_1 Z_2 e^2}{2\pi \epsilon_0 h v} \propto E^{-1/2}$

we get:

$$\boxed{\sigma(E) \propto e^{-bE^{-1/2}}}$$

with

$$b = \frac{\pi \mu_m^{1/2} Z_1 Z_2 e^2}{2^{1/2} \epsilon_0 h}$$

Also from estimate of De Broglie wavelength:

$$\boxed{\sigma(E) \propto \pi \lambda^2 \propto \pi \left(\frac{h}{p}\right)^2 \propto \frac{1}{E}} \quad \text{(B)}$$

↳ Both "together":

$$\boxed{\sigma(E) = \frac{S(E)}{E} \cdot e^{-bE^{-1/2}}}$$

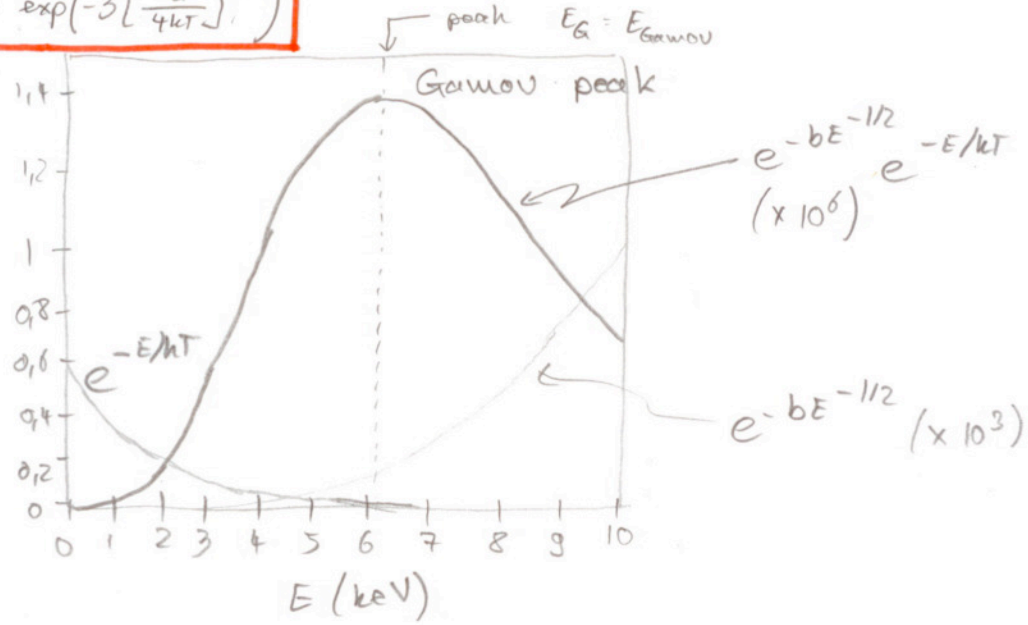
with $S(E)$ slowly varying in E.

→ Putting all together:

$$\boxed{R_{ix} = \left(\frac{2}{kT}\right)^{3/2} \frac{n_i n_x}{(\mu_m \pi)^{1/2}} \int_0^\infty S(E)\, e^{-bE^{-1/2}}\, e^{-E/kT}\, dE}$$

↳ numbers: $\boxed{R_{ix} = 6{,}48 \cdot 10^{-24} \frac{n_i n_x}{\mu_m Z_1 Z_2} S(E_0) \left(\frac{E_G}{4kT}\right)^{2/3} \exp\left(-3\left[\frac{E_G}{4kT}\right]^{1/3}\right) \frac{1}{m^3 s}}$

↳ $\boxed{R_{ix} \propto n_i n_x \cdot \left(\frac{E_G}{4kT}\right)^{2/3} \exp\left(-3\left[\frac{E_G}{4kT}\right]^{1/3}\right)}$



$e^{-E/kT}$: high energy wing of MB distribution

$e^{-bE^{-1/2}}$: comes from penetration probability

→ together: <u>strongly peaked curve!</u>

Maximum at $E_0 = \left(\frac{bkT}{2}\right)^{2/3}$

besides this continuum description, there may be <u>Resonances</u> and

<u>Electron screening</u> effects

(polarisation)

- <u>Power-law description</u>

often the rate R_{ix} can be described as power law:

$$R_{ix} \approx R_0 \cdot X_i X_x \rho^{\alpha'} T^{\beta}$$

if energy per reaction is ϵ_{ox}, then the total energy per unit mass is

$$\epsilon_{ix} = \frac{\epsilon_{ox}}{\rho} \cdot R_{ix} = \epsilon_{ox} X_i X_x \rho^{\alpha} T^{\beta}$$

$(\alpha = \alpha' - 1)$

- total energy released: sum over all reactions

$$\epsilon = \sum_x \epsilon_{ox} X_i X_x \rho^{\alpha} T^{\beta}$$

- the resulting luminosity gradient:

$$\boxed{dL = \epsilon\, dm}$$

where $\epsilon = \epsilon_{nuclear} + \epsilon_{gravity}$

Note $\epsilon_{gravity}$ can be negative if ✶ is expanding!

- with $dm = dM_r = \rho dV = 4\pi r^2 \rho \, dr$

$$\hookrightarrow \boxed{\frac{dL_r}{dr} = 4\pi r^2 \rho \epsilon}$$

- this is the last stellar structure equation. now we have everything to compute the structure of ✶ as function of mass, radius, composition, etc.

Nuclear Reaction Rates

- Stars consist mostly of protons, so let's start with

$$p + p \longrightarrow d + e^+ + \nu_e$$

this would be fast if it were not to need a <u>weak</u> interaction (to convert $p \rightarrow n$)

[$p \rightarrow n + e^+ + \nu$ needs 1,8 MeV, but this is paid back by binding energy of deuteron, which is 2,22 MeV]

[the p-p fusion rate is $5 \cdot 10^{13} \frac{1}{m^3 s}$
↳ a proton in center of Sun hangs around for $9 \cdot 10^9$ yr on average before it fuses with other proton]

[↳ this sets <u>nuclear</u> <u>time scale</u>]

- once there is d, it quickly reacts with another p to build ^3He:

$$p + d \longrightarrow {}^3He + \gamma$$

- now there are several ways to make ^4He:

I: ^3_2He fuses with ^3_2He to make $^4_2He + 2p$

$$^3_2He + ^3_2He \longrightarrow ^4_2He + 2p \qquad \boxed{\Delta E_{eff} = 26{,}2\ MeV}$$

II. or ^3_2He fuses with ^4_2He to ^7_4Be and

$$^3_2He + ^4_2He \longrightarrow ^7_4Be + \gamma$$
$$^7_4Be + e^- \longrightarrow ^7_3Li + \nu_e \qquad \boxed{\Delta E_{eff} = 25{,}2\ MeV}$$
$$^7_3Li + p \longrightarrow 2 \cdot ^4_2He$$

III or ^3_2He fuses again with ^4_2He to ^7_4Be, but now

$$^3_2He + ^4_2He \longrightarrow ^7_4Be + \gamma$$
$$^7_4Be + p \longrightarrow ^8_5B + \gamma \qquad \boxed{\Delta E_{eff} = 19{,}1\ MeV}$$
$$^8_5B \longrightarrow ^8_4Be + e^+ + \nu$$
$$^8_4Be \longrightarrow 2 \cdot ^4_2He$$

- the rate limiting reaction is the very first one:

$$S_{pp}(0) = 3{,}8 \cdot 10^{-22}\ keV\ barn \qquad (barn = 10^{-24}\ cm^2)$$

$$S_{pd}(0) = 2{,}5 \cdot 10^{-4}\ keV\ barn$$

this is 10^{18} times larger!

dominant for solar-type ∗'s

- in summary: **pp Chain**

$${}^1_1H + {}^1_1H \longrightarrow {}^2_1H + e^+ + \nu_e$$

$${}^2_1H + {}^1_1H \longrightarrow {}^3_2He + \gamma$$

Note: branching ratios differ from Carroll & Ostlie to Phillips!

69% [85%]

$${}^3_2He + {}^3_2He \longrightarrow {}^4_2He + 2\cdot {}^1_1H$$

PPI

31% [15%]

$${}^3_2He + {}^4_2He \longrightarrow {}^7_4Be + \gamma$$

99,7%

$${}^7_4Be + e^- \longrightarrow {}^7_3Li + \nu_e$$

$${}^7_3Li + {}^1_1H \longrightarrow 2\cdot {}^4_2He$$

PPII

0,3%

$${}^7_4Be + {}^1_1H \longrightarrow {}^8_5B + \gamma$$

$${}^8_5B \longrightarrow {}^8_4Be + e^+ + \nu_e$$

$${}^8_4Be \longrightarrow 2\cdot {}^4_2He$$

PPIII

effective energy gain:

$$\Delta E_{eff} = 69\% \cdot \frac{26{,}2\,MeV}{2} + 31\% \cdot 25{,}2\,MeV = \underline{16{,}9\,MeV}$$ (Carroll & Ostlie)

$$\Delta E_{eff} = 0{,}85 \cdot \frac{26{,}2}{2}\,MeV + 0{,}15 \cdot 25{,}2\,MeV = \underline{15\,MeV}$$ (Phillips)

- all together: $\epsilon_{pp} \approx 1{,}08 \cdot 10^{-5} \frac{erg/s}{cm^3\,g} \cdot \rho X^2 \cdot T^4 \cdot \underbrace{f_{pp}\,\psi_{pp}\,C_{pp}}_{\text{QM factors} \approx 1}$; X = H fraction

↳ $\boxed{\epsilon_{pp} \propto T_6^4 \quad \text{at} \quad T_6 \approx 15}$ $T_6 = 10^6 K$

↳ this is very modest T dependency.

- **CNO cycle for massive $*$'s**

- While the pp cycle can account for hydrogen burning in solar-type main sequence $*$'s it fails for massive $*$'s.

- as $L \propto M^{5,5}$ (for $* \neq \odot$) even a modest mass increase results in enormous luminosity gain. This is too much for the "modest" T^4-dependency of pp chain (recall: $T \propto \frac{M}{R}$)
  $\hookrightarrow$ something else must govern heat production in massive $*$'s

- steeper T-dependency required $\rightarrow$ larger coulomb barrier $\rightarrow$ must involve heavy elements $\rightarrow$ but as their abundance (at best!) is very low, they must be recycled to prolong H burning

- $\boxed{\text{CNO cycle with carbon, nitrogen, oxygen as catalysts!}}$

$$p + {}^{12}_{6}C \rightarrow {}^{13}_{7}N + \gamma \qquad S(0) = 1{,}5\,\text{keV barn}$$
$$\hookrightarrow {}^{13}_{6}C + e^+ + \nu_e$$
$$p + {}^{13}_{6}C \rightarrow {}^{14}_{7}N + \gamma \qquad S(0) = 5{,}5\,\text{keV barn}$$
$$p + {}^{14}_{7}N \rightarrow {}^{15}_{8}O + \gamma$$
$$\hookrightarrow {}^{15}_{7}N + e^+ + \nu_e \qquad S(0) = 3{,}3\,\text{keV barn}$$
$$p + {}^{15}_{7}N \rightarrow {}^{12}_{6}C + {}^{4}_{2}He \qquad S(0) = 78\,\text{keV barn}$$

the net result is $\boxed{4p \rightarrow {}^{4}_{2}He + 3\gamma + 2e^{+} + 2e\nu}$

$\boxed{\Delta E_{eff} = 23{,}8 \text{ MeV}}$

- just like in pp, there are several CNO chains

$${}^{12}_{6}C + {}^{1}_{1}H \rightarrow {}^{13}_{7}N + \gamma$$
$${}^{13}_{7}N \rightarrow {}^{13}_{6}C + e^{+} + \nu_e$$
$${}^{13}_{6}C + {}^{1}_{1}H \rightarrow {}^{14}_{7}N + \gamma$$
$${}^{14}_{7}N + {}^{1}_{1}H \rightarrow {}^{15}_{8}O + \gamma$$
$${}^{15}_{8}O \rightarrow {}^{15}_{7}N + e^{+} + \nu$$

99,96% | 0,04 %

$${}^{15}_{7}N + {}^{1}_{1}H \rightarrow {}^{12}_{6}C + {}^{4}_{2}He$$

$${}^{15}_{7}N + {}^{1}_{1}H \rightarrow {}^{16}_{8}O + \gamma$$
$${}^{16}_{8}O + {}^{1}_{1}H \rightarrow {}^{17}_{8}F + \gamma$$
$${}^{17}_{8}F \rightarrow {}^{17}_{8}O + e^{+} + \nu_e$$
$${}^{17}_{8}O + {}^{1}_{1}H \rightarrow {}^{14}_{7}N + {}^{4}_{2}He$$

- $\epsilon_{CNO} = 8{,}24 \cdot 10^{-24} \frac{\text{erg/s}}{\text{cm}^3 \text{g}^2} \cdot \rho \cdot XZ \cdot T_6^{\,19{,}9}$ at $T_6 = 15$

↳ very steep T dependency: $\boxed{\epsilon_{CNO} \propto T^{20} \text{ at } T_6 \approx 15}$

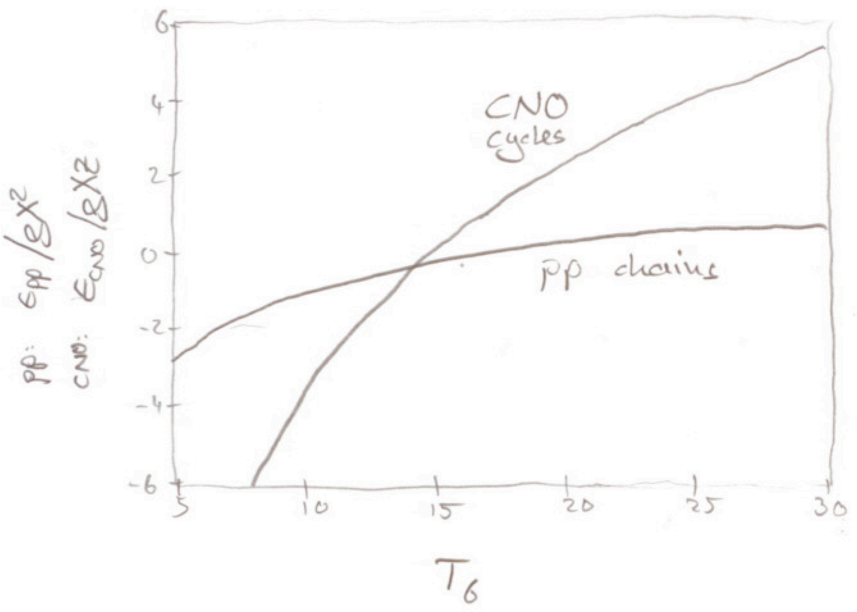
X = H fraction
Z = "metall" fraction

- Comparison pp & CNO:

- for Sun $(T_6 \approx 15,\ X = 0{,}7,\ Z \approx 0{,}02)$

$$\boxed{\epsilon_{pp} \approx 10 \times \epsilon_{CNO}}$$ → only 10% of ☉'s energy production from CNO

[Note: Phillips gives 2%]



- pp chains are lower in efficiency and energy yield than CNO (once CNO becomes possible)

$\epsilon_{pp} \approx 15 \ldots 17$ MeV
$\epsilon_{CNO} \approx 23 \ldots 27$ MeV
} for solar models

- Triple α-Process of Helium Burning

once the central density and temperature gets high enough, He burning can set in:

$${}^4_2He + {}^4_2He \longleftrightarrow {}^8_4Be$$

(unstable, decays back to $2\times {}^4_2He$ if not hit by other ${}^4_2He$)

$${}^8_4Be + {}^4_2He \longrightarrow {}^{12}_6C + \gamma$$

$\Delta E_{eff} = 7{,}3$ MeV

$\rightarrow$ $\boxed{\epsilon_{3\alpha} = \epsilon_{0,3\alpha} \cdot \rho^2 Y^3 \cdot T_8^{41}}$ Y = He fraction

$\rightarrow$ very steep T dependence: $\Delta T = 10\% \rightarrow \Delta L$ of 5000%!

- Carbon & Oxygen Burning

$${}^{12}_6C + {}^4_2He \longrightarrow {}^{16}_8O + \gamma$$

$${}^{16}_8O + {}^4_2He \longrightarrow {}^{20}_{10}Ne + \gamma$$

Production of α-elements!

Near $6\cdot10^8$ K: ${}^{12}_6C + {}^{12}_6C \longrightarrow$

${}^{16}_8O + 2\times {}^4_2He$	$\Delta E < 0$
${}^{20}_{10}Ne + {}^4_2He$	$\Delta E > 0$
${}^{23}_{11}Na + p^+$	$\Delta E > 0$
${}^{23}_{12}Mg + n$	$\Delta E < 0$
${}^{24}_{12}Mg + \gamma$	$\Delta E > 0$

Further oxygen burning at $T > 10^9$ K:

$$
{}^{16}_{8}O + {}^{16}_{8}O \longrightarrow \left\{
\begin{array}{lr}
{}^{24}_{12}Mg + 2 \times {}^{4}_{2}He & \Delta E < 0 \\
{}^{28}_{14}Si + {}^{4}_{2}He & \Delta E > 0 \\
{}^{31}_{15}P + p^{+} & \Delta E > 0 \\
{}^{31}_{16}S + n & \Delta E > 0 \\
{}^{32}_{16}S + \gamma & \Delta E > 0
\end{array}
\right.
$$

$\Delta E < 0$: energy is absorbed rather than released.

- Fusion goes up to ${}^{56}Fe$!

  ⟶ most stable element.

Summary of structure:

lower main sequence

$M < 1{,}5\, M_\odot$ (F0)

pp chain

$\epsilon_{pp} \propto T^4$

low T-dependency; less concentrated energy source
$\hookrightarrow$ small T-gradients

$\boxed{\text{Radiative CORE}}$

upper main sequence

$M > 1{,}5\, M_\odot$ (F0)

CNO cycle

$\epsilon_{CNO} \propto T^{20}$

in center: large flux & steep temperature gradient

$\boxed{\text{convective core}}$

$$T_C \quad > \quad 20 \cdot 10^6 \quad > \quad T_C$$

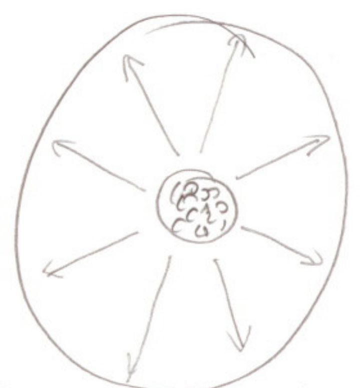
surface: H neutral, then ionization & rapid increase of opacity → steep T gradient

$\boxed{\text{convective envelope}}$

the smaller m, the further convective zone reaches into star.

surface hot & ionized
modest temperature gradient

$\boxed{\text{radiative envelope}}$



the larger m, the larger the convective core