9. Effects of Rotation

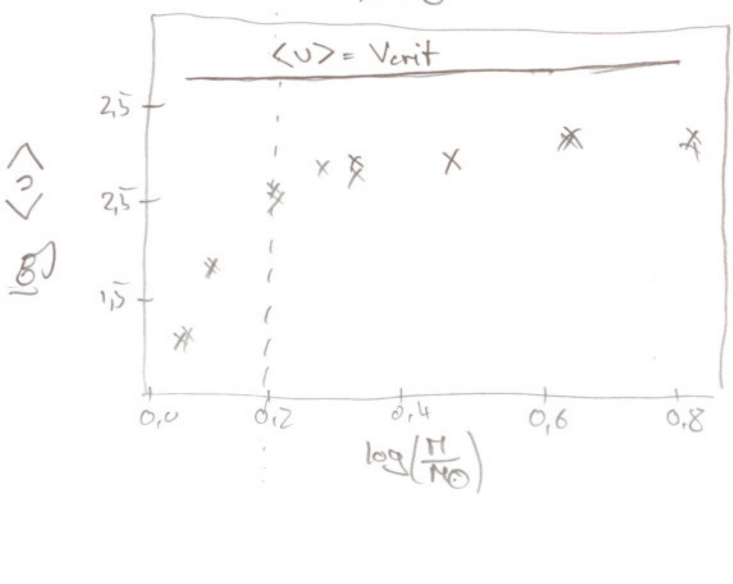
- Don Clayton: Rotation effects the evolution of a star in at least 2 ways:
  1. conservation of angular momentum during structural changes (contraction / expansion)
  2. onset of fluid circulation to maintain energy balance.

1. centrifugal forces render rotating star non-spherical
2. von Zeipels theorem & gravity darkening

- Let's start with observations:

$$\boxed{\begin{array}{l}\text{massive stars rotate rapidly} \\ \text{low-mass stars rotate slowly}\end{array}}$$

F,O = 1,5 $M_\odot$



Plot of average equatorial velocity for different stellar populations.

V_{crit} = break-up velocity $= \sqrt{\frac{GM}{R}}$

- Stars with masses below 1,5 $M_\odot$ (later than F0) have significant convection in envelope.

Recall:

lower main-sequence: core radiative
envelope convective

upper main sequence: core convective
envelope radiative

- Winds can carry away angular momentum. The amount of ang. mom. loss depends at radius at which wind decouples from stellar interior:

- Simple winds decouple near the photosphere. But: if wind couples to magnetic field it can be forced to corotate with the star well beyond the photosphere!

The B-field is rooted in stellar interior and as corotating wind moves beyond the photosphere it gains ang. mom. at the expense of the interior.

low-mass $*$'s have significant convection:
↳ relatively strong fields & winds
↳ strong ang. mom. loss

high-mass $*$'s: weaker fields → weaker mass loss.

- Implication: $\boxed{\text{Stars slow down during their evolution !}}$

- <u>von Zeipel's theorem</u>

- consider a $*$ that is in mechanical & thermal equilibrium
↳ rotation alters the shape of $*$
↳ equ's of hydrostatic equilibrium must include <u>centrifugal force</u>

- centrifugal acceleration can be associated with <u>centrifugal potential</u>:

Force: $\Omega^2 \cdot R \cdot \vec{e}_R = -\vec{\nabla}\psi$

↳ $\psi = -\int_0^R \Omega^2 R' dR' = -\frac{1}{2}\Omega^2 R^2$

for rigid rotator.

cylindrical coordinates: $(x,y,z) \rightarrow (R,\varphi,z)$

↳ effective potential:

$$\phi_{eff} = \phi + \psi = \phi - \frac{\Omega^2 R^2}{2}$$

↳ eqn of hydrostatic equilibrium:

$$\boxed{\frac{1}{\rho}\vec{\nabla} P = -\vec{\nabla}\phi_{eff} = -\vec{\nabla}\phi + \Omega^2 R\, \vec{e}_R} \quad *$$

$= \vec{g}_{eff}$ = acceleration due to local effective gravity.

also: Poisson: $\boxed{\vec{\nabla}^2 \phi = 4\pi G \rho}$ note, only true gravity, ϕ, appears here.

- consider "level" surfaces of constant ϕ_{eff}:

- $\vec{\nabla}\phi_{eff}$ evaluated on such surface is always $\perp$ to surface

- consider $d\phi_{eff} = d\vec{r} \cdot \vec{\nabla}\phi_{eff}$, with $d\vec{r}$ being tangent vector ↳ $d\phi_{eff} = 0$

- back in (*) ⟶ $\frac{1}{\rho} d\vec{r} \cdot \vec{\nabla} P = 0$ as well

↳ $\boxed{P = \text{const. on equipotential surface}}$

- for barotropic $*$: $P = P(\rho)$

$$\Longrightarrow \boxed{P = P(\Phi_{eff})} \ \& \ \boxed{\Phi_{eff} = \Phi_{eff}(P)}$$

with (*): $\frac{1}{\rho} = \frac{d\Phi_{eff}}{dP}$ = const. on equipot. surf.

$$\Longrightarrow \boxed{\rho = \rho(\Phi_{eff})}$$

- now consider chemically homogeneous star (μ = const.)

$$P = \frac{\rho k T}{\mu m_p} \Longrightarrow \boxed{T = T(\Phi_{eff})}$$

- So, the only "thing" not constant on equipot. surf is effective acceleration $\vec{g}_{eff}$. it is always $\perp$ to surface, but its magnitud varies over surface!

- consider 2nd constraint: <u>Thermal equilibrium</u>

  - thermal balance: $\frac{dL}{dr} = 4\pi r^2 \rho \epsilon$
  - in terms of radiative flux:

$$\vec{\nabla} \cdot \vec{F} = \rho \epsilon$$

  - $F_{rad} = -\frac{c}{\kappa \rho} \cdot \frac{dP_{rad}}{dr} = -\frac{c}{\kappa \rho} \cdot \frac{4}{3} a T^3 \frac{dT}{dr}$

for spherical star.

for the rotating star, rephrase in terms of $d\Phi_{eff}$:

$$\boxed{\vec{F} = -\frac{4ac}{\varkappa\rho} T^3 \cdot \frac{dT}{d\Phi_{eff}} \cdot \vec{\nabla}\Phi_{eff}}$$

if the star is thermally stable, then

$$\boxed{\vec{\nabla}\cdot\vec{F} = 0}$$

but, this is <u>NOT</u> fullfilled:

take $\boxed{\vec{F} = K\cdot\vec{\nabla}\Phi_{eff}}$ then

$$\vec{\nabla}\cdot\vec{F} = \vec{\nabla}K\cdot\vec{\nabla}\Phi_{eff} + K\cdot\vec{\nabla}^2\Phi_{eff}$$

$$= \frac{dK}{d\Phi_{eff}}\left(\vec{\nabla}\Phi_{eff}\cdot\vec{\nabla}\Phi_{eff}\right) + K\,\vec{\nabla}^2\Phi_{eff}$$

– our assumption is $T = $ const, $P = $ const, $\rho = $ const on Φ_{eff} surface $\Rightarrow$ $\frac{dK}{d\Phi_{eff}} = 0$ on surface

$\Rightarrow$ $\vec{\nabla}\cdot\vec{F} = K\,\vec{\nabla}^2\Phi_{eff} = K\cdot\left[4\pi G\rho - 2\Omega\right]$

this cannot vanish every where !

$\Rightarrow$ ϟ <u>von Zeipel's paradox.</u>

- <u>Solution:</u> not in equilibrium → $\boxed{\text{MERIDIONAL MOTIONS}}$

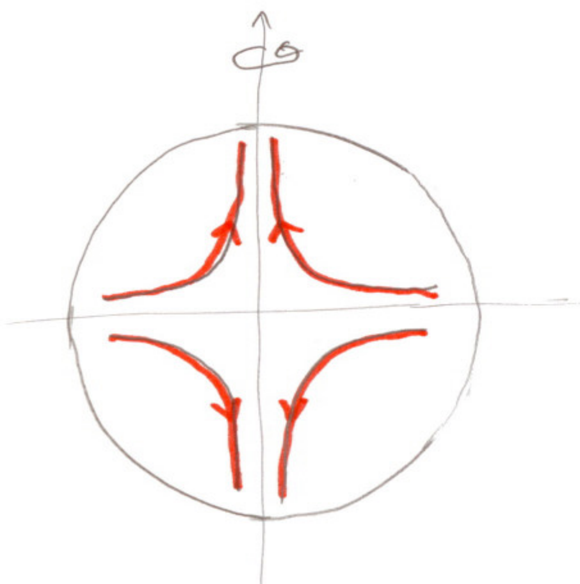
  - surface area per unit solid angle greater on equator than on pole

  - in equilibrium: same flux transported through larger surface area as through smaller one

  - temperature drops locally on equator to compensate for that and rises on poles.

  - because of bouancy effects warmer gas close to rotational axis begins to <u>rise</u> and colder gas at the equator begins to <u>fall</u>

- meridional circulation will <u>mix</u> the star



- star is <u>hotter = brighter</u> on poles and <u>colder = darker</u> on equator

  ↳ $\boxed{\text{gravity darkening}}$ $\boxed{T_{eff} \propto g^{1/4}}$

- Note: the time scale for meridional circulation is long:

critical parameter $\beta = \frac{E_{rot}}{|E_{grav}|} = \frac{R^3 \Omega^2}{GM}$

$E_{rot} = \Omega^2 R^2 M$

$E_{grav} = \frac{GM^2}{R}$

$\hookrightarrow$ $\boxed{t_{circ} \approx \frac{t_{KH}}{\beta} = \left(\frac{GM}{R^3 \Omega^2}\right) \cdot t_{KH}}$

- for rapid rotators $\beta \approx 0,1$

$\hookrightarrow$ $t_{circ} \approx 10^2 \, t_{KH} \approx 10^9$ yr : This is longer than main sequence lifetime!

$\hookrightarrow$ mixing is not efficient

- for slow rotators $\beta \lesssim 0,01$

$\hookrightarrow$ $t_{circ} \approx 10^4 \, t_{KH} \approx 10^{11}$ yr : still longer than MS lifetime.

- NB: this meridional motion may also be induced by tidal perturbations in close binary systems.

10. Magnetic Fields in Stars

↳ A brief deviation into Dynamo Theory

- relation between current j and electric and magnetic field $\vec{E}$, $\vec{B}$:

$$\vec{j} = \sigma \left(\vec{E} + \frac{\vec{v}}{c} \times \vec{B} \right)$$

σ = electrical conductivity

- chance of B:

$$\boxed{\frac{\partial \vec{B}}{\partial t} = -c\, \vec{\nabla} \times \vec{E}}$$

induction equation

$$\hookrightarrow \frac{\partial \vec{B}}{\partial t} = \vec{\nabla} \times \vec{v} \times \vec{B} - \frac{c}{\sigma} \vec{\nabla} \times \vec{j}$$

$$= \vec{\nabla} \times \vec{v} \times \vec{B} + \eta \vec{\nabla}^2 \vec{B}$$

with $\eta = \frac{4\pi c^2}{\sigma}$ = diffusion coeff.

- if we decompose all fields into a mean part and a fluctuating part:

$$\vec{j} = \bar{j} + \vec{j}'$$
$$\vec{E} = \bar{E} + \vec{E}'$$
$$\vec{B} = \bar{B} + \vec{B}'$$

then we get: $\bar{j} = \sigma \left(\bar{E} + \frac{1}{c} \bar{v} \times \bar{B} + \frac{1}{c} \overline{\vec{v}' \times \vec{B}'} \right)$

$$\hookrightarrow \frac{\partial \bar{B}}{\partial t} = \vec{\nabla} \times (\bar{v} \times \bar{B}) + \eta \vec{\nabla}^2 \bar{B} + \vec{\nabla} \times \mathcal{E}$$

$\mathcal{E}$ = mean electromotive force

$= \overline{\vec{v}' \times \vec{B}'} = \langle \vec{v}' \times \vec{B}' \rangle$

- assume $\mathcal{E}$ comes from turbulent motions!

↳ if $\vec{v}'$ and $\vec{B}'$ (the fluctuating parts) are completely uncorrelated then

$$\overline{\vec{v}' \times \vec{B}'} = \langle \vec{v}' \rangle \times \langle \vec{B}' \rangle = 0$$

- assume $\vec{v} = 0$:

– what remains is:

$$\boxed{\frac{\partial \overline{B}}{\partial t} = \eta \vec{\nabla}^2 \overline{B}}$$

this is a diffusion equation: the magn. field "diffuses" away: it decays in the absence of dynamo action!

– time scale: τ : $\frac{\overline{B}}{\tau} \approx \eta \frac{\overline{B}}{L^2}$

↳ $\tau \approx \frac{L^2}{\lambda} \approx \frac{4\pi\sigma L^2}{c^2}$

Earth: $L \approx 3 \cdot 10^8$ cm $\sigma \approx 10^{16}$ esu → $\tau \approx 3 \cdot 10^5$ yr

Sun: $L \approx 5 \cdot 10^{10}$ cm $\sigma \approx 10^{17}$ esu → $\tau \approx 10^{11}$ yr

Galaxy: $L \approx 3 \cdot 10^{20}$ cm $\sigma \approx 10^{10}$ esu → $\tau \approx 3 \cdot 10^{23}$ yr

- assume $\mathcal{E} = 0$ & $\eta = 0$

$$\boxed{\frac{\partial \bar{B}}{\partial t} = \vec{\nabla} \times (\vec{v} \times \bar{B})}$$

ideal MHD limit ($\sigma \to \infty$) infinite conductivity

↳ this leads to flux freezing!

the $\bar{B}$-field is compressed just like density

$$\frac{d\bar{B}}{dt} = -\vec{B}(\vec{\nabla}\cdot\vec{v}) \quad \& \quad \frac{d\rho}{dt} = -\rho\,\vec{\nabla}\cdot\vec{v}$$

with $\frac{d}{dt} = \frac{\partial}{\partial t} + \vec{v}\cdot\vec{\nabla}$

↳ $B = |\bar{B}| = f\cdot\rho$ ⟶ scales linearly with ρ

- Dynamo amplification of field comes from electromotive force $\mathcal{E}$:

Statistical turbulence theory allows for the ansatz:

$$\mathcal{E} = \langle \vec{v}' \times \vec{B}' \rangle = \alpha \cdot \bar{B} - \eta_T \vec{\nabla} \times \bar{B}$$

with η_T being $\frac{1}{3}\langle \vec{v}'\cdot\vec{v}'\rangle\cdot\tau$ the turbulent resitivity.

it turns out $\eta_T \gg \eta$ (molecular resistivity) if $\alpha \gg \eta_T$ $\Rightarrow$ dominates

$\Rightarrow$ α-Ω dynamo

- Note: dynamos can only amplify preexisting fields $\Rightarrow$ $\boxed{\text{need a seed field}}$