Stellar Astronomy and Astrophysics (SS07):

Exercise 2 (for May 24, 2007)

1. Chemical time scale:

Assuming that 10 eV could be released by every atom in the Sun through chemical reactions, estimate how long the Sun could shine at its current rate ($L_\odot = 6.96 \cdot 10^{33}\,\mathrm{erg\,s^{-1}}$) through chemical processes alone. For simplicity, we assume that the Sun is composed entirely of hydrogen.

2. Temperature to overcome Coulomb barrier (classically):

(a) Show that the root-mean-square speed of the Maxwell-Boltzmann velocity distribution

$$n(v)\mathrm{d}v = 4\pi n \left(\frac{m}{2\pi kT}\right)^{3/2} v^2 \exp\left(\frac{-mv^2}{2kT}\right) \mathrm{d}v$$

in thermodynamic equilibrium is $v_{\mathrm{rms}} = \sqrt{\frac{3kT}{m}}$, with T being the thermodynamic temperature, $k = 1.38 \cdot 10^{-16}\,\mathrm{erg\,K^{-1}}$ being the Boltzmann constant, and m being the mass of the particles.

(b) Taking into consideration the Maxwell-Boltzmann velocity, what temperature would be required for two protons to collide if quantum mechanical tunneling is neglected? Assume that nuclei having velocities ten times the root-mean-square value for the Maxwell-Boltzmann distribution can overcome the Coulomb barrier, if we assume that the particles collide if they get as close as $1\,\mathrm{fm} = 10^{-15}\,\mathrm{m}$. The mass of a proton is $m_{\mathrm{p}} = 1.67 \cdot 10^{-24}\,\mathrm{g}$. Compare the answer to the central temperature of the sun.

(c) What is the ratio of the number of protons moving with $v = 10\,v_{\mathrm{rms}}$.

(d) Assuming that the Sun is pure hydrogen, estimate the number of hydrogen nuclei in the Sun. Could there be enough protons moving with ten times the rms value to account for the Sun's luminosity.

3. Polytropic spheres:

$$\frac{1}{\xi^2}\frac{\mathrm{d}}{\mathrm{d}\xi}\left[\xi^2 \frac{\mathrm{d}D_n}{\mathrm{d}\xi}\right] = -D_n^n$$

is the Lane-Emden equation for a polytropic index n ($P = K\rho^\gamma = K\rho^{(n+1)/n}$).
Recall that $\rho(r) = \rho_c[D_n(r)]^n$, where $0 \le D_n \le 1$, $\lambda_n = \left[(n+1)\left(\frac{K\rho_c^{(1-n)/n}}{4\pi G}\right)\right]^{1/2}$, and $r = \lambda_n \xi$.
The surface is defined by $D_n(\xi_{\mathrm{surf}}) = 0$, and the inner boundary is $\mathrm{d}D_n/\mathrm{d}\xi = 0$ at $\xi = 0$.

(a) Show that the $n = 0$ polytrope has a solution given by

$$D_0(\xi) = 1 - \frac{\xi^2}{6}, \text{ with } \xi_{\mathrm{surf}} = \sqrt{6}.$$

(b) Show that the $n = 1$ polytrope has a solution given by

$$D_1(\xi) = \frac{\sin\xi}{\xi}, \text{ with } \xi_{\mathrm{surf}} = \pi.$$

(c) Show that the $n = 5$ polytrope has a solution given by

$$D_5(\xi) = (1 + \xi^2/3)^{-1/2}, \text{ with } \xi_{\rm surf} \to \infty.$$

Is the total mass finite?

(d) Plot the density structure for $n = 0, 1$, and 5 (ρ_n/ρ_c vs. r/λ_n).