Stellar Structure & Evolution

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1 Introduction

Stars are huge self-gravitating spheres of gas

- stars are **very massive**

  they must reach very high internal temperature
  to generate stability against gravitational collapse

- stars have **no walls**

  they must have outer boundaries which communicate
  all excess energy from their interior to space
  (they radiate energy away)

In the end, this means, they **never** are in the **true equilibrium** state
(however, we often assume quasi-equilibrium).

Stars are huge self-gravitating spheres of gas, with nuclear
reactions providing the energy source for quasi-equilibrium

$\hookrightarrow$ Structure and evolution governed by equations of hydrodynamics

Our goal here: Understanding structure on **main sequence**;
i.e. structure of star in **quasi-equilibrium**;

$\hookrightarrow$ We seek **time-independent** solutions of hydro equations

Fundamental assumption: stars are perfect spheres

$\hookrightarrow$ we consider simple, 1D spherical symmetry

$\hookrightarrow$ this simplifies our equations.

1.1 Equations of hydrodynamics

Recall from theoretical astrophysics course.
Derived from Boltzmann equation via moment-building.

- **continuity of mass:**

$$\frac{d\varrho}{dt} = \frac{\partial\varrho}{\partial t} + (\vec{v}\cdot\vec{\nabla})\varrho = -\varrho\,\vec{\nabla}\cdot\vec{v}$$

- **Navier-Stokes: momentum equation**

$$\frac{d\vec{v}}{dt} = \frac{\partial\vec{v}}{\partial t} + (\vec{v}\cdot\vec{\nabla})\vec{v} = -\frac{1}{\varrho}\,\vec{\nabla}P - \vec{\nabla}\phi + \eta\,\vec{\nabla}^2\vec{v} + (\xi + \frac{\eta}{3})\vec{\nabla}(\vec{\nabla}\cdot\vec{v})$$

- **energy equation:**

$$\begin{aligned}\frac{d\epsilon}{dt} = \frac{\partial\epsilon}{\partial t} + (\vec{v}\cdot\vec{\nabla})\epsilon &= T\frac{ds}{dt} - \frac{P}{\varrho}\vec{\nabla}\cdot\vec{v} \\ &= T\frac{ds}{dt} - \frac{P}{\varrho^2}\frac{d\varrho}{dt} \\ &= T\frac{ds}{dt} - P\frac{dV}{dt}\end{aligned}$$

- **Poisson equation:**

$$\vec{\nabla}^2\phi = 4\pi G\varrho$$

- Closure equation → **equation of state:**

$$P = fct(\varrho, T)$$

- **EOS of ideal gas:**

$$\begin{aligned}PV &= NkT \\ P &= \frac{k\varrho T}{\mu m_p}\end{aligned}$$

$$\begin{aligned}k &= 1.38\cdot 10^{-16}\,\mathrm{erg/K} \\ \mu &= \text{mean molecular weight} \\ m_p &= \text{mean mass of proton} = 1.67\cdot 10^{-24}\,\mathrm{g}\end{aligned}$$

[be careful, this changes throughout star]

- **Polytropic EOS:**

$$P = K\varrho^{\gamma}$$

$$\gamma = \text{polytropic index} = \frac{\text{specific heat at constant } P}{\text{specific heat at constant } V}$$

$$\gamma = c_p/c_v \;; \quad c_p = c_v + R$$

for monoatomic gas $\gamma = \frac{5/2\,R}{3/2\,R} = 5/3$

- **isothermal EOS:**

$$P = K\varrho = c_s^2\varrho$$

isothermal $\gamma = 1$

in general thermodynamic state of the gas depends on **balance** between **heating** and **cooling** processes
$\hookrightarrow$ the effective γ can have any value > 0 !

in general $\gamma = 1 + \dfrac{d\ln T}{d\ln\varrho}$

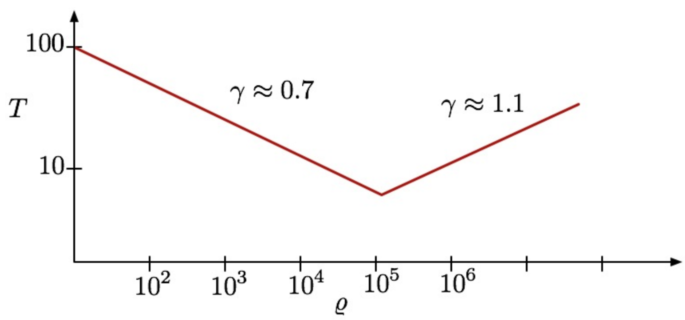


Figure 1.1:

Gas cools more strongly as it gets compressed $\Lambda \propto \varrho^2$ $T\downarrow$	coupling to dust $\hookrightarrow$ gas heats up again $T\uparrow$

Our goal is to determine, how ϱ, P, T, ϵ change with position in star.

- **Fundamental assumption:** stars are spheres

  $\hookrightarrow$ **1D spherical symmetry**

$$\left.\begin{aligned}
&(x,y,z) \;\rightarrow\; (r,\theta,\phi) \\
&\vec{\nabla}_{\vec{x}} = \vec{e}_x \frac{\partial}{\partial x} + \vec{e}_y \frac{\partial}{\partial y} + \vec{e}_z \frac{\partial}{\partial z} \\
&\vec{\nabla}_{\vec{r}} = \vec{e}_r \frac{\partial}{\partial r} + \frac{\vec{e}_\theta}{r} \frac{\partial}{\partial \theta} + \frac{\vec{e}_\phi}{r\sin(\theta)} \frac{\partial}{\partial \phi}
\end{aligned}\right\}\text{Spherical coordinates}$$

spherical symmetry:

$$\frac{\partial}{\partial \theta} \rightarrow 0 \quad \frac{\partial}{\partial \phi} \rightarrow 0$$

$\hookrightarrow$ only **radial** dependency remains.

We seek all "interesting" variables as function of

Radius (r)

$$\frac{dP}{dr} = ? \quad \frac{dM_r}{dr} = ?$$
$$\frac{dL_r}{dr} = ? \quad \frac{dT}{dr} = ?$$

or equivalently (as there is a one-to-one relation between r and M_r) as function of

Mass (M_r)

$$\frac{dP}{dM_r} = ? \quad \frac{dr}{dM_r} = ?$$
$$\frac{dL_r}{dM_r} = ? \quad \frac{dT}{dM_r} = ?$$

M_r	=	Mass within radius r
r	=	Radius from center outwards
P	=	Pressure
L	=	Luminosity
T	=	Temperature

2 Hydrostatic Balance

- Consider star in quasi-static equilibrium

  $\hookrightarrow$ no acceleration

- in Navier-Stokes:

$$\frac{d\vec{v}}{dt} = 0$$

$$\frac{d\vec{v}}{dt} = -\frac{1}{\varrho}\vec{\nabla}P - \vec{\nabla}\Phi = 0$$

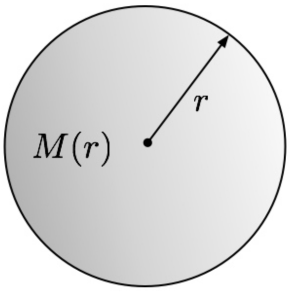
$$\frac{dP}{dr} = -\varrho\frac{GM(r)}{r^2} \tag{2.1}$$

force due to pressure gradient = force due to gravity

we use

$$\vec{\nabla}\phi = \frac{d\phi}{dr} = \frac{d}{dr}\left[-\frac{GM(r)}{r}\right] = \frac{GM(r)}{r^2}$$

- Enclosed mass or: equivalence of r and $M(r)$



$$\begin{aligned} M_r &= M(r) = \int_0^r 4\pi\varrho(r')r'^2\,dr' \\ &= \text{mass interior to } r \end{aligned}$$

$$\frac{dM_r}{dr} = 4\pi\varrho(r)r^2 \quad\Longleftrightarrow\quad \frac{dr}{dM_r} = \frac{1}{4\pi\varrho r^2} \tag{2.2}$$

using mass as independent variable

$$\frac{dP}{dr} = -\varrho(r)\,\frac{G\,M(r)}{r^2} \quad\overset{(2.2)}{\Longleftrightarrow}\quad \frac{dP}{dM_r} = -\frac{G\,M_r}{4\pi\,r^4}$$

From equation (2.1) follows: star must be hot in the center in order to hold it up against contraction by sufficient pressure gradients.

as first approach say

$$\frac{dP}{dr} \rightarrow \frac{P}{R} \qquad (R = \text{radius of star})$$

and assume ideal gas law $P = \dfrac{\varrho\, kT}{\mu\, m_p}$

$$\begin{aligned} \frac{P}{R} &\approx \langle\varrho\rangle \frac{G\,M}{R^2} \\ \frac{1}{R}\frac{\langle\varrho\rangle\, kT_c}{\mu\, m_p} &\approx \langle\varrho\rangle \frac{G\,M}{R^2} \qquad \longrightarrow \qquad kT_c \approx \mu\, \frac{G\,M\,m_p}{R} \end{aligned}$$

$\hookrightarrow$ if star shrinks, it must get hotter for fixed M

$$R \downarrow \qquad T_c \uparrow$$

$\hookrightarrow$ of mean molecular weight increases, star must get hotter in counter for fixed M and R

$$\mu \uparrow \qquad T_c \uparrow$$

(important for stellar evolution as H burns into He)

3 Dynamical Time

So far we assume quasi-equilibrium, i. e.

$$\frac{d\vec{v}}{dt} = 0$$

What happens, if this is not the case?

- Navier-Stokes:

$$\frac{d\vec{v}}{dt} = \underbrace{-\frac{1}{\varrho}\vec{\nabla}P}_{\text{drop pressure force}} - \frac{G\,M}{r^2}\vec{e}_r$$

$$\frac{d\vec{v}}{dt} = -\frac{G\,M(r)}{r^2}\,\vec{e}_r$$

- Collapse on a dynamical time scale:

$$v \approx \frac{R}{\tau_{dyn}}; \quad M(r) \approx \langle\varrho\rangle\,R^3; \quad \frac{d}{dt} \approx \frac{1}{\tau_{dyn}}$$

$$\frac{R}{\tau_{dyn}^2} \approx \frac{\langle\varrho\rangle G R^3}{R^2}$$

$$\tau_{dyn} \approx \frac{1}{\sqrt{G\varrho}} \approx \frac{1\,\mathrm{hour}}{(\varrho/1\,\mathrm{g\,cm^{-3}})^{1/2}}$$

- This is the shortest time scale available to the star.

4 Virial Balance

- Instead of looking at the balance of **forces**, we can compare energies

  $\hookrightarrow$ virial theorem

  Have we derive it by integrating equation (2.1)

- Take $\frac{dP}{dr} = -\varrho \frac{G\,M(r)}{r^2}$ and multiply by $4\pi r^3\,dr$ on both sides and integrate:

$$\begin{aligned}\int_0^R 4\pi r^3 \frac{dP}{dr}\,dr &= -\int_0^R \frac{GM(r)}{r} \underbrace{4\pi\varrho r^2\,dr}_{dM_r} \\ &= -\int_0^R \frac{GM(r)}{r}\,dM_r = +\omega\end{aligned}$$

LHS: integrate by parts:

$$\int_0^R 4\pi r^3 \frac{dP}{dr}\,dr = \underbrace{4\pi r^3 P(r)\Big|_0^R}_{=0} -3\int_0^R P(r)\underbrace{4\pi r^2\,dr}_{dV}$$

$$\begin{bmatrix} R\text{: } P(r) & =0 \\ 0\text{: } R & =0 \end{bmatrix}$$

$$= -3\int_0^R P\,dV \overset{!}{=} -2U$$

- From the theoretical astrophysics lecture we know, the “full” virial theorem for self-grav-gas is

$$\frac{1}{2}\ddot{I} = 2I + 2U + \omega$$

  in equilibrium $\ddot{I} = 0$ (tensor of inertia).

- If there is no bulk motion or rotation, $T = 0$

$$2U + \omega = 0$$
$$U = \frac{3}{2}\int P\,dV$$
$$\omega = \frac{1}{2}\int \phi\varrho\,dV$$

- **Note:** $U = \frac{3}{2}\int P\,dV$ is **NOT** the internal energy

$$U_{int} = \frac{3}{2}NkT \quad !$$

(not all the internal energy is able to do work!)

internal energy density: $\epsilon = \frac{3}{2}nkT$; Pressure: $P = nkT$

we know:

$$\begin{aligned}\gamma \cdot \epsilon &= \frac{3}{5}\cdot\frac{3}{2}nkT \\ &= \frac{5}{2}nkT \\ &= \left(\frac{3}{2}+1\right)nkT \\ &= \epsilon + P\end{aligned}$$

$$P = (\gamma - 1)\epsilon \quad \text{with } \gamma = \frac{c_p}{c_V}$$

thus:

$$\begin{aligned}U = \frac{3}{2}\int P\,dV &= \frac{3}{2}\int(\gamma-1)\epsilon\,dV \\ &= \frac{3}{2}(\gamma-1)\underbrace{\int \epsilon\,dV}_{\text{internal energy } U_{int}} \\ &= \frac{3}{2}(\gamma-1)U_{int}\end{aligned}$$

$\hookrightarrow$ virial theorem:

$$\begin{aligned}2U + \omega &= 0 \\ 3(\gamma-1)U_{int} + \omega &= 0\end{aligned}$$

- total energy:

  $$E_{tot} = U_{int} + \omega$$

  together:

  $$E_{tot} = U_{int} - 3(\gamma - 1)U_{int} = -(3\gamma - 4)U_{int}$$

  because $U_{int} > 0$, the equilibrium system is bound, i.e. $E < 0$, only of

  $$\gamma > \frac{4}{3}$$

  this defines a critical polytrope for gravitational collapse to occur.

  What happens as γ approaches $4/3$?

  consider:

  $$\begin{aligned} \ddot{I} &= 2U + \omega \\ &= 3(\gamma - 1)U_{int} + \omega \\ &= 3(\gamma - 1)(E - \omega) + \omega \\ &= 3(\gamma - 1)E - (3\gamma - 4)\omega \end{aligned}$$

  $$E = \frac{\gamma - 4/3}{\gamma - 1}\omega + \frac{1}{3(\gamma - 1)}\ddot{I}$$

  for $\gamma \to 4/3$ the first term on RNS vanishes and if we manage to keep $E < 0$

  then $\ddot{I} < 0$; implying contraction.

Also seen via linear stability analysis [Carroll and Ostlie, 2006, §14.3] for $\gamma < 4/3$ exponentially growing

modes $\longrightarrow$ collapse.

4.1 Application of Virial Theorem: Kelvin - Helmholtz timescale

- for a star, the total energy is always negative

  $E_{tot} = -(3\gamma - 4)U_{int} \quad \& \quad E_{tot} = \frac{3\gamma-4}{3\gamma-3}\omega$

  if $\gamma > \dfrac{4}{3}$

$$
\begin{aligned}
w &= -\int\limits_0^R \frac{GM(r)}{r}\, 4\pi\varrho(r) r^2\, dr \\
&= -\int\limits_0^R \frac{4}{3} G\pi\langle\varrho\rangle r^3 \cdot 4\pi\langle\varrho\rangle \cdot r\, dr \\
&= -\frac{16}{15} G\pi^2 \langle\varrho\rangle^2 R^5 = -\frac{3}{5}\frac{GM^2}{R}
\end{aligned}
$$

  assume constant mean density $\langle\varrho\rangle = \dfrac{M}{4/3\pi R^3}$

- for $\gamma = \dfrac{4}{3}$

$$E_{tot} = \frac{5-4}{5-3}\,\omega = \frac{1}{2}\,\omega = -\frac{3}{10}\frac{GM^2}{R}$$

- a star that loses (internal) energy must **contract**

  E_{tot} gets smaller (more negative) if R gets smaller !

- luminosity of star

$$L = -\frac{dE}{dt} = +\frac{3}{10}\frac{GM^2}{R}\frac{\dot{R}}{R}$$

  constant luminosity requires **constant** contraction $\dot{R}$!

- **characteristic timescale**

$$\dot{R} = \frac{R}{t_{KH}} \quad \Rightarrow \quad t_{KH} = \frac{3}{10}\frac{GM^2}{RL}$$

  this is the **Kelvin - Helmholz timescale**

  for the Sun:

$$
\begin{aligned}
L_\odot &= 4 \cdot 10^{33}\ \mathrm{erg/s} = 4 \cdot 10^{33}\ \mathrm{g\,cm^2/s^3} \\
M_\odot &= 2 \cdot 10^{33}\ \mathrm{g} \\
G &= 6.67 \cdot 10^{-8}\ \mathrm{cm^3/g\,s^2} \\
R_\odot &= 7 \cdot 10^{10}\ \mathrm{cm}
\end{aligned}
$$

$$t_{KH} = 9.5 \cdot 10^{14}\,\mathrm{s} = 30\,\mathrm{Myr}$$

this timescale is short compared to age of Earth.
⇒ gravitational contraction cannot be energy source of the Sun !

- **Note:** star is always in hydrostatic balance as it contracts, because

$$t_{KH} \gg \tau_{dyn} \approx 1 \text{ hour}$$

- a star radiates, i.e. loses energy continuously

  it can only escape contraction if a new source of energy becomes available to compensate for the radiative losses at surface

- What can that be?

$$E_{grav} \approx \frac{GM^2}{R} \;\rightarrow\; \frac{E_{grav}}{M} = \frac{GM}{R} \approx 10^{15} \text{ erg/g}$$

chemitry: $\dfrac{5eV}{atom} \approx 5 \cdot 10^{12}$ erg/g

nuclear reactions: $\dfrac{7eV}{m_p} \approx 7 \cdot 10^{18}$ erg/g

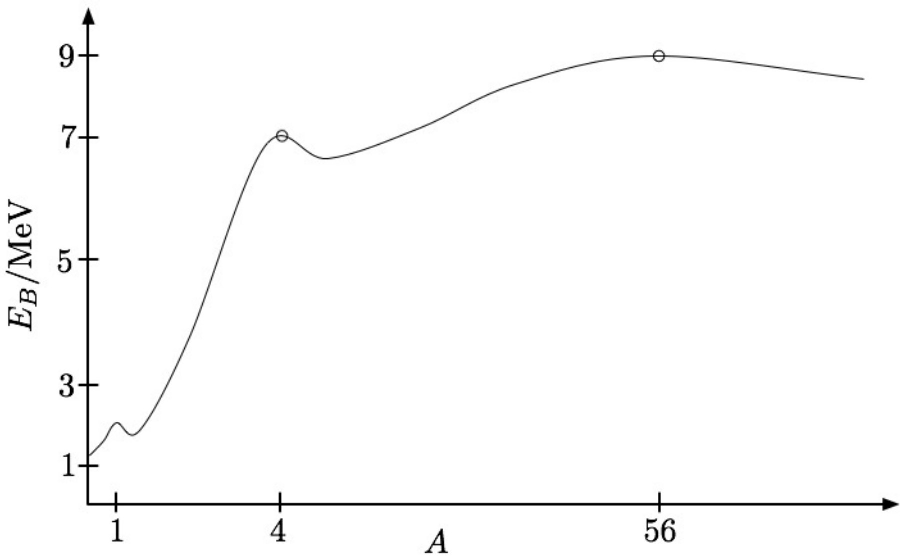


Figure 4.1:

5 Equation of state for radiation dominated gases

- Recall the polytropic EOS:

$$P = K\,\varrho^{\gamma}$$

- recall also the ideal gas law:

$$PV = NkT = \frac{R}{\mu}T$$

- recall first law of thermodynamic:

$$dQ = dU + PdV$$

dQ = heat change
dU = change of internal energy

$$U = u\,V \quad \& \quad P = \frac{1}{3}u$$

energy density for radiation is

$$u = \frac{4\pi}{c}\int B_\nu(T)\,d\nu$$

- entropy change: $ds = T^{-1}\,dQ$

$$\begin{aligned} ds &= \frac{V}{T}\frac{du}{dT}\,dT + \frac{u}{T}\,dT + \frac{1}{3}\frac{u}{T}\,dV \\ &= \frac{V}{T}\frac{du}{dT}\,dT + \frac{4}{3}\frac{u}{T}\,dV \end{aligned}$$

$$\text{build} \quad \left.\frac{\partial s}{\partial T}\right|_V = \frac{V}{T}\frac{du}{dT} \quad \& \quad \left.\frac{\partial s}{\partial V}\right|_T = \frac{4}{3}\frac{u}{T}$$

$$\text{build} \quad \frac{\partial^2 s}{\partial V \partial T} = \frac{1}{T}\frac{du}{dT} \quad \& \quad \frac{\partial^2 s}{\partial V \partial T} = -\frac{4}{3}\frac{u}{T^2} + \frac{4}{3T}\frac{du}{dT}$$

because

$$\frac{\partial^2 s}{\partial V \partial T} - \frac{\partial^2 s}{\partial T \partial V} \quad \longrightarrow \quad \frac{1}{3T}\frac{du}{dT} = \frac{4}{3}\frac{u}{T^2}$$

$$\frac{du}{u} = \frac{4}{T}\,dT \quad \longrightarrow \quad \ln u = 4 \ln T + \ln a$$

with $\ln a$ =constant of integration

Stefan-Boltzmann law:

$$U(T) = aT^4$$

Radiation pressure:

$$P_{rad} = \frac{1}{3}U = \frac{a}{3}T^4$$

the total energy of the radiation-influenced gas is

$$U = \frac{3}{2}\frac{R}{\mu}T + aT^4V = c_vT + aT^4V$$

total pressure:

$$P = P_{gas} + P_{rad} = \frac{R}{\mu}\frac{T}{V} + \frac{a}{3}T^4$$

Now define the ratio of gas pressure to total pressure:

$$\beta = \frac{P_{gas}}{P} \qquad \text{with} \quad 0 \le \beta \le 1$$

$$\text{then} \quad P_{gas} = \frac{\varrho kT}{\mu m_p} = \beta P \qquad \text{for gas pressure} \tag{5.1}$$

$$\text{and} \quad P_{rad} = \frac{a}{3}T^4 = (1-\beta)P \qquad \text{for radiation pressure} \tag{5.2}$$

We are looking for a polytropic EOS that is independent of T, we combine both expressions to eliminate T.

- in (5.1): $T = \beta P(\varrho k)^{-1}\,\mu m_p$ insert in (5.2):

$$\frac{a}{3}\left(\frac{\beta P \mu m_p}{\varrho k}\right)^4 = (1-\beta)P$$

- then: $P = K\varrho^{4/3}$

  with $K = \left(\dfrac{3\,(1-\beta)}{a}\right)^{1/3}\left(\dfrac{k}{\beta\mu m_p}\right)^{4/3}$

- The EOS of radiation dominated systems has a polytropic index $\gamma = 4/3$ (this corresponds to a polytrop of index $n = (\gamma-1)^{-1}$)

- This is similar to relativistic, degenerate objects
  (clear, because photons are relativistic)
- Also radiation dominated stars are close to the stability limit!

  We can estimate, when P_{rad} dominates over P_{gas}:
- recall from hydrostatic balance:

$$kT \sim \frac{GMm_p}{R} \qquad \text{set} \quad \mu = 1$$

- we have

$$\frac{P_{rad}}{P_{gas}} \sim \frac{aT^4 m_p}{\varrho kT}$$

we know (theoretical astrophysics):

$$a = \frac{8}{15}\frac{\pi^5 k^4}{c^3 h^3} = 7.5 \cdot 10^{-15} \frac{\text{erg}}{\text{cm}^3\text{K}^{-4}} \approx \frac{k^4}{(\hbar c)^3}$$

$$\begin{aligned}
\frac{P_{rad}}{P_{gas}} &\sim \frac{k^4}{(\hbar c)^3}\frac{T^4 m_p}{\varrho kT} = \frac{(kT)^3}{(\hbar c)^3}\frac{m_p}{\varrho} = \underbrace{\left(\frac{GMm_p}{R}\right)^3 \frac{R^3}{M}\frac{m_p}{(\hbar c)^3}}_{\text{independent of R!}} \\
&\sim \underbrace{\left(\frac{Gm_p^2}{\hbar c}\right)^3}_{\alpha_a = \frac{Gm_p^2}{\hbar c} = 6 \cdot 10^{-39}} \left(\frac{M}{m_p}\right) \\
&\sim \alpha_a^2 \left(\frac{M}{m_p}\right)^2 \qquad \text{(only function of mass)}
\end{aligned}$$

critical mass:

$$\begin{aligned}
M &\approx \alpha_G^{-3/2}\, m_{photon} \\
&\approx 2 \cdot 10^{57}\, m_p \\
&\approx 4 \cdot \mathrm{M}_\odot
\end{aligned}$$

(Real Answer: $M \geq 60\,\mathrm{M}_\odot$ or so $\rightarrow$ we dropped to many pre-factors)

6 Equation of state for degenerate gas

- for non-relativistic particles: $P = K\varrho^{5/3}$
- for relativistic particles: $P = K'\varrho^{4/3}$

Derivation a la Padmanabhan, p.2 + 3:

$$P = \frac{1}{3}\int_0^\infty n(\epsilon)p(\epsilon)v(\epsilon)d\epsilon$$

$$\begin{aligned} p &= \gamma m v \\ \epsilon &= (\gamma - 1)mc^2 \\ \gamma &= \left(1 - \frac{v^2}{c^2}\right)^{-1/2} \end{aligned}$$

- rate of momentum transfer from particles with energy ϵ
- 1/3 comes from isotropy
- the system is called ideal, when the kinetic energy dominates over their interaction energy

then: $$P = \frac{1}{3}\int_0^\infty n\epsilon \left(1 + \frac{2mc^2}{\epsilon}\right)\left(1 + \frac{mc^2}{\epsilon}\right)^{-1} d\epsilon$$

Non-relativistic limit: $mc^2 \gg \epsilon$

$$P \approx \frac{2}{3}\int_0^\infty n\epsilon d\epsilon = \frac{2}{3}\langle n\epsilon\rangle = \frac{2}{3}u$$

Relativistic limit: $mc^2 \ll \epsilon$ (or particles are massless: photons)

$$P \approx \frac{1}{3}\int_0^\infty n\epsilon d\epsilon = \frac{1}{3}\langle n\epsilon\rangle = \frac{1}{3}u$$

with u being energy density.

- kinematic energy ϵ for particle with momentum p and mass m:

$$\epsilon = \sqrt{p^2c^2 + m^2c^4} - mc^2 = \begin{cases} \frac{p^2}{2m} & p \ll mc \\ pc & p \gg mc \end{cases}$$

$$[P] = \mathrm{g\,cm/s}$$

for **DEGENERATE GAS**:

- uncertainty principle: gives minimum volume a free particle can occupy in phase space

$$\Delta x \Delta p = h$$

- if particle separation, on average, given by number density, $\Delta x = n^{-1/3}$, then $p_F = h\, n^{1/3}$ (exclusion principle!)

- Because for isotropic gas

$$P_{\text{degenerate}} = \frac{1}{3}U_F = \frac{1}{3}\epsilon\varrho \qquad \text{[NR.]}$$

$$P_{\text{degenerate}} = \frac{2}{3}U_F = \frac{2}{3}\epsilon\varrho \qquad \text{[Rel.]}$$

we can write in

Non-Relativistic case $\quad P_{deg} \propto \dfrac{p_F^2 n}{m} \quad \Rightarrow \quad P_{deg}^{NR} = K^{NR}\varrho^{5/3}$

Relativistic case $\quad P_{deg} \propto p_F\, n \quad \Rightarrow \quad P_{deg}^{ER} = K^{ER}\varrho^{4/3}$

- For degenerate relativistic particles that are compressed, the pressure increase is smaller since also the **"apparent" mass** is increased.
The effect is as if the mean molecular weight is **increased**, the medium becomes **more compressible**! (EOS becomes softer)

Note: the fully relativistic degenerate case has the same scaling as radiation EOS!

7 Structure of Polytropic Spheres

- We go back to the equations of hydrostatic balance:

$$\frac{dP}{dr} = -\varrho \frac{GM(r)}{r^2} \tag{7.1}$$

$$\frac{dMr}{dr} = 4\pi \varrho r^2 \tag{7.2}$$

These are the first of the desired stellar structure equation:

We note, the system can be solved **without** reference to any energy or energy transport equation!

- We take the radial derivative of equation (7.1):

$$\frac{d}{dr}\left(\frac{r^2}{\varrho}\frac{dP}{dr}\right) = -G\,\frac{dM_r}{dr}$$

and use equation (7.2) for the RHS

$$\frac{d}{dr}\left(\frac{r^2}{\varrho}\,\frac{dP}{dr}\right) = -4\pi \varrho r^2 \qquad \text{(this is a Poisson equation)}$$

or

$$\frac{1}{r^2}\frac{d}{dr}\left(\frac{r^2}{\varrho}\,\frac{dP}{dr}\right) = -4\pi\, G\varrho \tag{7.3}$$

Because in hydrostatic equilibrium pressure gradients follow gravitational potential gradients:

$$\frac{1}{\varrho}\frac{dP}{dr} = -\frac{d\phi}{dr}$$

We can also write:

$$\frac{1}{r^2}\frac{d}{dr}\left(r^2\,\frac{d\phi}{dr}\right) = +4\pi G\varrho$$

We solve equation (7.3) by using a polytropic EOS:

$$P = K\varrho^\gamma$$

we get

$$\frac{\gamma}{r^2}K\frac{d}{dr}\left[r^2\rho^{\gamma-2}\,\frac{d\varrho}{dr}\right] = -4\pi G\varrho$$

- Often people use a different notation:

  Define $\gamma = (n+1)/n$ where **n =polytropic index**

$$\left(\frac{n+1}{n}\right)\frac{K}{r^2}\frac{d}{dr}\left[r^2\varrho^{(1-n)/n}\frac{d\varrho}{dr}\right] = -4\pi G\varrho$$

- We simplify by expressing density in terms of scaling factor: **central density** ϱ_c and a dimensionless function $D_n(r)$:

$$\varrho(r) = \varrho_c[D_n(r)]^n \quad \text{with } 0 \leq D_n(r) \leq 1$$

  we get:

$$(n+1)\left(\frac{K\varrho_c^{(1-n)/n}}{4\pi G}\right)\frac{1}{r^2}\left[r^2\,\frac{dD_n(r)}{dr}\right] = -D_n^n(r)$$

- simplify further:

$$\lambda_n = \left[(n+1)\left(\frac{K\varrho_c^{(1-n)/n}}{4\pi G}\right)\right]^{1/2}$$

  and introduce dimensionless variable ξ

$$r = \lambda_n \xi$$

  and we finally get

$$\frac{1}{\xi^2}\frac{d}{d\xi}\left(\xi^2\frac{dD_n}{d\xi}\right) = -D_n^n \qquad \textbf{Lane-Emden equation} \tag{7.4}$$

- Solving equation (7.4) for the dimensionless function $D_n(\xi)$ in terms of ξ for specific index n leads directly to the **density profile** $\boldsymbol{\varrho_n(r)}$.
  The polytropic EOS $\mathbf{P_n(r) = K\varrho_n^{(n+1)/n}(r)}$ provides the pressure gradients.
  If we assume ideal gas law and a constant ratio of P_{gas} and P_{rad}, we can also obtain the **temperature profile** $T(r)$.

- To solve equation (7.4), we need to specify **two** boundary conditions.
  (1) We want the central density to be finite: $\frac{d\varrho}{dr} \to 0$ as $r \to 0$

$$\frac{dD_n}{d\xi} = 0 \quad \text{at} \quad \xi = 0$$

  (2) We want the density go to zero at the surface: $\varrho(R) = 0$

$$D_n(\xi) = 0 \quad \text{at surface} \quad \xi = \xi_1$$

- we also require the normalization $D_n(0) = 1$, so that ϱ_c = central density
- total mass: $M = 4\pi \int\limits_0^R \varrho r^2 \, dr$

  radius of (7.3): $R = \lambda_n \xi_1$

$$\begin{aligned} M &= 4\pi \int\limits_0^{\xi_1} \lambda_n^2 \xi^2 \varrho_c D_n^n(\xi) \, d(\lambda_n \xi) \\ &= 4\pi \lambda_n^3 \varrho_c \int\limits_0^{\xi_1} \xi^2 D_n^n \, d\xi \end{aligned}$$

  this integral can be **solved directly**.

  OR: we note that $\xi^2 D_n^n = -\frac{d}{d\xi}\left[\xi^2 \frac{dD_n}{d\xi}\right]$

$$M = -4\,\pi \lambda n^3 \varrho_c \, \xi_1^2 \frac{dD_n}{d\xi}\bigg|_{\xi_1}$$

  that means, the total mass can be derived by evaluating the derivative of D_n at the surface ξ_1.

- Limitations: equation (7.4) contains **NO** information about energy generation and energy transport in the star.
- There are **three** analytic solutions

$$\begin{aligned} n = 0: &\quad D_0(\xi) = 1 - \frac{\xi^2}{6} && \text{with} \quad \xi_1 = \sqrt{6} \\ n = 1: &\quad D_1(\xi) = \frac{\sin\xi}{\xi} && \text{with} \quad \xi_1 = \pi \\ n = 5: &\quad D_5(\xi) = \left(1 + \frac{\xi^2}{2}\right)^{-1/2} && \text{with} \quad \xi_1 \to \infty \end{aligned}$$

  although $\xi_1 \to \infty$, the total mass remains finite for D_5.
  This is no longer the case for $\mathbf{n > 5}$!

  thus physically allowed: $0 \leq n \leq 5$

- The discussion of polytrops was originally motivated by adiabatic EOS.

$$\left(\gamma = \frac{5}{3} \stackrel{\wedge}{=} n = 1.5\right)$$

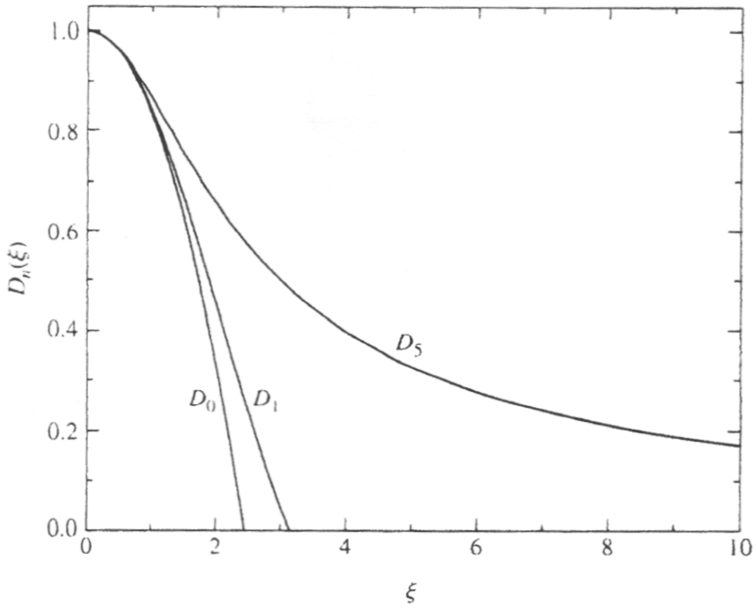


Figure 7.1: The analytic solution to the Lane-Emden equation. $D_0(\xi),\ D_1(\xi),\ D_5(\xi)$

This holds very well for degenerate, non-relativistic objects, such as white dwarfs, here also (as discussed below):

$$\gamma = \frac{5}{3} \ \&\ n = \tfrac{3}{2}$$

- $n = 3$ is the "Eddington" standard model of a star in **radiative equilibrium**.

  We have seen before that in this case

$$P = K\varrho^{4/3}$$

7.1 Application of polytropes

(A) White Dwarfs:

- **non-relativistic electron degeneracy**
- $\gamma = 5/3 \quad \Rightarrow \quad n = 3/2$

(B) Chandrasekhar:

- limit for **relativistic** electron degeneracy
- $\gamma = 4/3 \quad \Rightarrow \quad n = 3$

$\hookrightarrow$ existence of **critical mass**!

The polytropic EOS with Lane-Emden equation leads to a relation between mass and radius:

$$\left(\frac{GM}{M_n}\right)^{n-1}\left(\frac{R}{R_n}\right)^{3-n} = \textbf{Constant} \tag{7.5}$$

with constants:

$$M_n = -\xi_1^2 \left.\frac{dD_n}{d\xi}\right|_{\xi_1} 4\pi\lambda_n^3\rho_c$$

$$R_n = \lambda_n \xi_1$$

********** **********

For $\mathbf{n = 3}$ (relativistic degeneracy or radiation dominated star)
the **dependence** on **radius r vanishes**!

There is only one possible value for the mass given in terms of K:

$$M_{crit} = 4\pi M_3 \left(\frac{K}{\pi G}\right)^{3/2}$$

only this mass satisfies hydrostatic equilibrium

substitution: of constants and parameters:

$$M_{crit} = \frac{5.83}{\mu_e^2}\,\mathrm{M}_\odot$$

for hydrogen- poor stars we get $\mu_e = 2$ and

$$M_{crit} = 1.46\,\mathrm{M}_\odot \quad \textbf{Chandrasekhar mass}$$

Note: This limiting mass can also be derived from the **virial theorem**:

- recall the condition for hydrostatic equilibrium:

$$\begin{aligned}\frac{dP}{dr} &= -\varrho(r)\,\frac{GM(R)}{r^2}\\ \frac{P}{R} &\approx \langle\varrho\rangle\,\frac{GM}{R^2} \longrightarrow P \approx \varrho\,\frac{GM}{R} = n\,\frac{GMm_p}{R}\mu\end{aligned}$$

  with $\langle\varrho\rangle = \mu m_p n$

- Fermi energy:

$$p_F = \hbar\,k_F = \hbar\,n^{1/3} = \hbar\left(\frac{M}{\mu m_p R^3}\right)^{1/3} \qquad \text{(Fermi momentum)}$$

$$E_F = np_F c = \hbar c\left(\frac{M}{\mu m_p}\right)^{\frac{1}{3}}\frac{n}{R} \qquad \text{(Fermi energy)}$$

  with $n \approx \dfrac{1}{R^3}\dfrac{M}{\mu m_p}$

  in balance:

$$p \approx \frac{GM\mu m_p}{R}n \approx \hbar c\left(\frac{M}{\mu m_p}\right)^{1/3}\frac{n}{R}$$

$$G^3M^3(\mu m_p)^3 \approx (\hbar c)^3\frac{M}{\mu m_p}$$

$$\frac{M^2}{m_p^2} \approx \frac{(\hbar c)^3}{G^3 m_p^6} \qquad \text{with } \mu \approx 1$$

$$M \approx m_p\frac{1}{\alpha_G^{3/2}} \qquad \alpha_G = \frac{Gm_p}{\hbar c} = 6\cdot 10^{-39}$$

  similar to the radiation dominated case, there is a critical mass for relativistic electron degeneracy!

  ⇒ **Chandrasekhar mass**

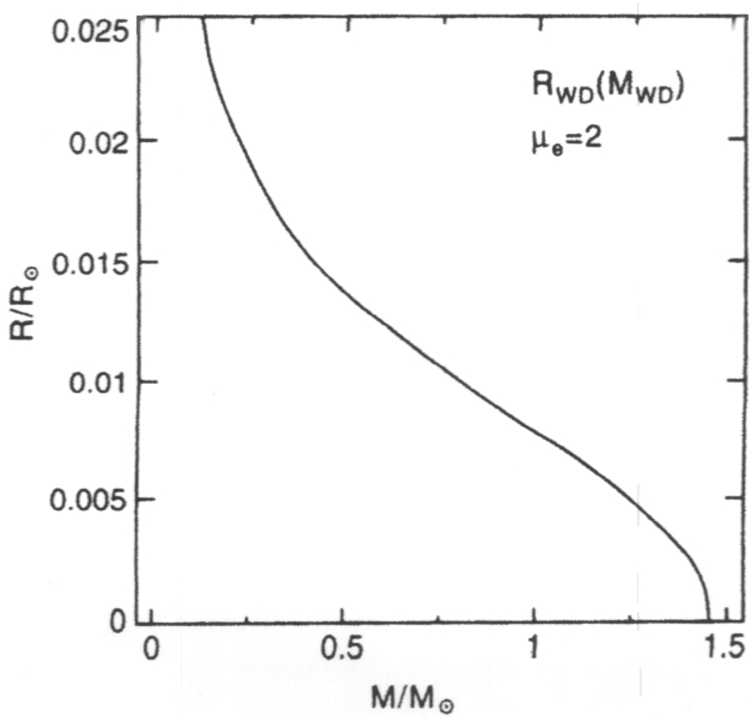


Figure 7.2: Mass-Radius relationship for white dwarfs

now consider $\mathbf{n = 1.5}$ **polytropes**: (non-relativistic degenerate EOS)

- examples: White Dwarfs
- from 7.4 we get:

$$\begin{aligned} R &\propto M^{-1/3} \\ \langle \varrho \rangle &\propto MR^{-3} \propto M^2 \end{aligned}$$

- **sequence**: let´s start with a non-relativistic degenerate (low-mass) white dwarf
- if we add mass, material in the center gets denser, because the radius shrinks.
- n will vary throughout the star; in the interior, you may reach $n = 3$
- as you add up more and more mass, more of the white dwarfs will be relativistic
- eventually everything is $n = 3$;

  BUT: stability only for one mass: $\mathbf{M \approx 1.46}$ M$_\odot$
- if the mass is larger ⇒ **collapse** sets in!

8 Energy Transport

There are **three** mechanisms to transport energy in the stellar interior:

- **RADIATION:**

  Energy transport by photons

  - photons can be absorbed and reemitted in different wavelength and direction
  - coupling to matter via **opacity** κ.

- **CONVECTION:**

  Energy is transported via bulk motion in the fluid

  - buoyant mass element carry excess heat outwards, while cooler elements more towards the center

- **CONDUCTION:**

  Heat transport via collisions between gas particles

  - usually is not important in stars

8.1 Radiative Temperature Gradient 1

- Radiation pressure gradient:

  $$\frac{dP_{rad}}{dr} = -\langle\kappa\rangle \frac{\varrho}{c} F_{rad}$$

  where F_{rad} = outwards radiative flux, ϱ = density, c = speed of light and the material constant κ = opacity.

- The opacity κ describes coupling between radiation and matter.

- Recall: for blackbody radiation, the radiation pressure is 1/3 of the energy density:

  $$P_{rad} = \frac{1}{3} u$$

- Opacity = absorption coefficient:
  - in radiative transfer equation $dI_v = -\kappa_v \varrho I_v\, ds$, the intensity at frequency v is attenuated along a distance ds by the factor $\kappa_v \varrho$.
  - recall: formal integration gives

$$I_v = I_{v,0} e^{-\kappa \varrho s} \tag{8.1}$$

  - recall also: for scattered photons, the characteristic distance l is the **mean free path** of photons:

$$l = \frac{1}{\kappa_v \varrho} = \frac{1}{n\sigma_v}$$

    with σ_v = interaction cross section (e.g. via Thomson scattering)
  - definition of optical depth:

$$d\tau_v = -\kappa_v \varrho ds$$

    along a light ray
  - formal integration turns into

$$I_v(s) = I_v(0) e^{-\kappa \varrho s} \stackrel{\wedge}{=} I_v(\tau) = I_v(0) e^{-\tau}$$

optical depth = number of mean free paths through the medium.

$\tau_v \ll 1$: **Optically thin Limit**

(light with frequency v passes through medium essentially unattenuated.)

$\tau \ll 1$: **Optically thick Limit**

(light gets “caught” in the medium, you can’t see through.)

- Sources of opacity:
  1. **BOUND - BOUND - TRANSITION** (excitation de-excitation)
  2. **BOUND - FREE ABSORPTION**
  3. **FREE - FREE ABSORPTION** (scattering)
  4. **ELECTRON SCATTERING**

     scattering by **free** electron: **Thomson**

     scattering by **loosely-bound electron**

     Compton if $\lambda \gg R_{atom}$
     Rayleigh if $\lambda \ll R_{photon}$

8.2 Rosseland Mean Intensity

- It is often useful to employ an opacity that is **averaged** over all wavelength (or frequencies) to produce a function that depends only on **composition, density and temperature.**
- the most common approach is to compute a **harmonic mean** weighted with the **Planck function:**

$$\frac{1}{\langle\kappa\rangle} = \int_0^\infty \frac{1}{\kappa_v}\frac{dB_v(T)}{dT}dv \left(\int_0^\infty \frac{\partial B_v(T)}{\partial T}dv\right)^{-1}$$

  Rosseland mean opacity

- there is no analytic solution, however there are analytic approximations (fits) to the various process that may contribute to $\langle\kappa\rangle$:
  - bound-free opacity:

$$\langle\kappa_{bf}\rangle = 4.32 \cdot 10^{22}\, \frac{g_{bf}}{t} Z(1+X)\frac{\varrho}{T^{3.5}}\ \mathrm{cm^2g^{-1}}$$

  - free-free opacity:

$$\langle\kappa_{ff}\rangle = 3.68 \cdot 10^{19} g_{ff}(1-Z)(1+X)\frac{\varrho}{T^{3.5}}\ \mathrm{cm^2g^{-1}}$$

  - electron scattering:

$$\langle\kappa_{es}\rangle = 0.2\,(1+X)\ \mathrm{cm^2g^{-1}}$$

  - H^- :

$$\langle\kappa_{\mathrm{H}^-}\rangle \approx 7.9 \cdot 10^{-33}\left(\frac{Z}{0.02}\right)\varrho^{1/2}T^9\ \mathrm{cm^2g^{-1}}$$

- **total Rosselland mean:**

$$\langle\kappa\rangle = \left\langle \underbrace{\kappa_{bb}}_{(A)} + \underbrace{\kappa_{bf}}_{(B)} + \underbrace{\kappa_{ff}}_{(C)} + \underbrace{\kappa_{es}}_{(D)} + \underbrace{\kappa_{\mathrm{H}^-}}_{(E)} \right\rangle$$

(A) There is no analytic fit to (A) because of **complexity** of contributions from different lines. (best from Tables)

$(B)+(C)$ have the well known form of **Kramer´s opacity law:**

$$\langle\kappa\rangle = \kappa_0 \frac{\varrho}{T^{3.5}}$$

X, Y, Z - mass fractions of H, He and the rest $(X+Y+Z=1)$

Gaunt factors g_{bf} and g_{ff} are quantum-mechanical corrections:

$$g \approx 1$$

$t =$ guillotine factor, describes decrease of opacity after atoms have been ionized.

$$t = 1 \ldots 100$$

(D) Because electron scattering is wavelength independent $\langle\kappa\rangle_{es}$ has a particularly simple form.

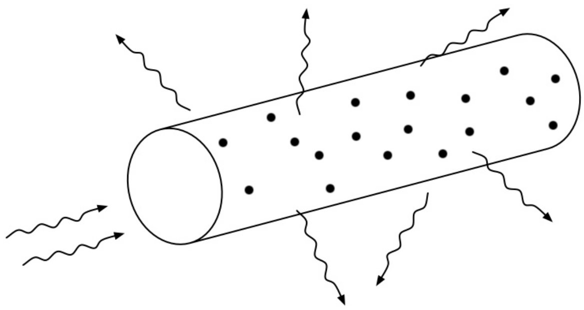
(E) in cooler stars such as the Sun bound-free and free-free transitions of H^- contribute significantly to the opacity in the range

$3000\,\mathrm{K} \lesssim T \lesssim 6000\,\mathrm{K}$ and $10^{-10}\frac{\mathrm{g}}{\mathrm{cm}^3} \lesssim \varrho \lesssim 10^{-5}\frac{\mathrm{g}}{\mathrm{cm}^3}$

κ_{H^-} increases with T as T^9!

Note:

(D) Electron scattering:



$$dI_v = -\kappa_v \varrho I_v ds$$

simple geometry

$$\kappa = \frac{\sigma_e n_e}{\varrho}\,[\mathrm{cm}^2\mathrm{g}^{-1}]$$

the cross section is
the classical electron radius:

$$\begin{aligned}\sigma_e &= \frac{8\pi}{3}\left(\frac{e^2}{mc^2}\right)^2 \\ &= 0.6652 \cdot 10^{-24}\,\mathrm{cm}^2\end{aligned}$$

- as long as photon energy is below $m_e c^2$, the opacity of Thomson scattering is independent of v.
- number of e^- comes from ionization of hydrogen mostly

(C) - electron can absorb photon only when ion is present $j\varrho \propto n_e n_i$

- for computing κ_{ff} turn problem around and study bremsstrahlung
- integrate over all electrons (velocity) states

  integrate Maxwell-Boltzmann distribution function

$$4\pi j_w \varrho dw = \frac{2\pi}{3}\frac{Z_e^2 e^6}{m_e c^2}\left(\frac{2\pi}{m_e kT}\right)^{1/2} n_e n_i e^{-\hbar\omega/kT} d\omega$$

$\hookrightarrow$ yields $j \propto T^{-1/2}$! (note change in sign)

- together:

$$j\varrho \propto n_e n_i T^{+1/2}$$

  fully ionized H

$$n_e = n_i = \frac{\varrho}{m_i} = n$$

- in equilibrium:

$$dI = (-\langle\kappa\rangle\varrho I + j\varrho)\, ds \stackrel{!}{=} 0$$

$$\langle\kappa_{ff}\rangle = \frac{j}{I}$$

- in LTE:

$$\begin{aligned} I &= S = B(T) && \text{Planck Function (integrated)} \\ &= aT^4 && \text{Stefan-Boltzmann law} \end{aligned}$$

$$\langle\kappa_{ff}\rangle \propto \frac{nT^{1/2}}{T^4} \propto nT^{-3.5} \qquad \textbf{Kramers opacity law}$$

(B) similar for bf: $\kappa_{bf} \propto nT^{-3.5}$!

8.3 Radiative Temperature Gradient 2:

- Now that we have studied possible sources of opacity in the stellar interior, we can turn back to computing the temperature stratification in the star.
- radiation pressure gradient:

$$\frac{dP_{rad}}{dr} = -\frac{\langle\kappa\rangle\varrho}{c}F_{rad}$$

- from $P_{rad} = \frac{1}{3}U_{rad} = \frac{a}{3}T^4$ we also get

$$\frac{dP_{rad}}{dr} = \frac{4}{3}aT^3\frac{dT}{dr}$$

- combining both we get far $\frac{dT}{dr}$:

$$\frac{dT}{dr} = -\frac{3}{4}\frac{\langle\kappa\rangle\varrho}{acT^3}F_{rad}$$

- We can now express the radiative flux in terms of the local radiative luminosity at radius r.

$$F_{rad} = \frac{L_r}{4\pi r^2}$$

- thus:

$$\frac{dT}{dr} = -\frac{3}{4}\frac{\langle\kappa\rangle\varrho}{acT^4}\frac{L_r}{4\pi r^2}$$

  temperature gradient for radiative transport.

8.3.1 Example: A simple mass-luminosity relation based on Thomson scattering (for hot, high-mass stars)

$$L \propto M^3\mu^5$$

- Stars are hot in center:

$$kT_c \approx \frac{\mu G M m_p}{R} \tag{8.2}$$

  cold at surface → heat transport

- energy flux

$$\begin{aligned} F_{rad} &= -\frac{c}{\langle\kappa\rangle\varrho}\frac{dP_{rad}}{dr} \\ &= -\frac{c}{\langle\kappa\rangle\varrho}\frac{4}{3}aT^3\frac{dT}{dr} \end{aligned}$$

- Electron scattering (Thomson):

$$\langle\kappa\rangle = \frac{\sigma_e n_e}{\varrho} \quad \text{with } \sigma_e = \frac{8\pi}{3}\left(\frac{e^2}{mc^2}\right)^2$$

$$F_{rad} = -\frac{c}{n_e\sigma_e}\frac{4}{3}aT^3\frac{dT}{dr}$$

- with equation (8.2):

$$\begin{aligned} F_{rad} &\approx -\frac{4}{3}\frac{ca}{n_e\sigma_e}T_e^3\frac{T}{R} \\ &\approx -\frac{4}{3}\frac{ca}{n_e\sigma_e}\left(\frac{\mu G M m_p}{kR}\right)^4\frac{1}{R} \end{aligned}$$

- luminosity:

$$\begin{aligned} L &\approx 4\pi R^2\, F \\ &\approx \frac{16}{3}\pi R^2\frac{ac}{\sigma_e}\frac{\mu m_p}{\varrho}\left(\frac{\mu G M m_p}{kR}\right)^4\frac{1}{R} \end{aligned}$$

drop all constants and use $\varrho \propto MR^{-3}$

$$L \propto R^2\frac{R^3}{M}\frac{M^4}{R^5}\mu^5$$

$$L \propto M^3\mu^5$$

- Plugging in all constants, gives a pre-factor not that far from $L_\odot = 4\cdot 10^{33}\,\text{erg/s}$ for $M_\odot = 2\cdot 10^{33}\,\text{g}$.

- **Dependencies:**

  When Thomson scattering dominates opacity ($M > \mathrm{M}_\odot$)
  L is **independent** of stellar radius and proportional to M^3.

  Composition determines luminosity, also:

  pure H:

$$P = \frac{2kTm_p}{\varrho} \rightarrow \mu = \frac{1}{2}$$

  pure He:

$$\begin{aligned} P &= n_\alpha kT + 2n_\alpha kT = 3n_\alpha kT \\ &= \frac{3}{4}\frac{\varrho kT}{m_p} \rightarrow \mu = \frac{4}{3} \end{aligned}$$

  ($L \uparrow$ for He)

8.3.2 Example: Mass luminosity relation for low-mass Stars (where Kramer´s opacity dominates)

- Again:

$$F_{rad} = -\frac{4}{3}aT^4 \frac{dT}{dr}$$

- Use $\langle\kappa\rangle \propto \varrho T^{-3.5}$ and $T_c \approx \frac{GM\mu m_p}{kR}$

$$\begin{aligned} F_{rad} &\propto -\frac{4}{3}cs \underbrace{\varrho^{-2}}_{\sim \frac{M}{R^3}^{-2}} \overbrace{T^{6.5}}^{T_c^{6.5}} \underbrace{\frac{dT}{dr}}_{\approx \frac{T_c}{R}} \\ &\propto \frac{R^6}{M^2}\frac{T_c^{7.5}}{R} \\ &\propto \frac{R^6}{M^2}\frac{M^{7.5}}{R^{8.5}} \propto \frac{M^{5.5}}{R^{2.5}} \end{aligned}$$

- the luminosity: $L \approx 4\pi R^2\, F_{rad}$

$$L \propto R^2 \frac{M^{5.5}}{R^{2.5}} \rightarrow L \propto \frac{M^{5.5}}{R^{0.5}}$$

- Low-mass stars have very strong mass dependence

$$L \propto M^{5.5} \quad (R = \text{constant})$$

and luminosity also depends on stellar radius

$$L \propto R^{-0.5} \quad (\text{for } M = \text{constant})$$

8.4 Pressure Scale Height

- before we turn to convention transport, it is useful to define a characteristic length-scale associated with p pressure gradients:

Pressure Scale Height

$$\frac{1}{H_p} = -\frac{1}{P}\frac{dP}{dr}$$

The negative accounts for the fact that $P(r)$ decreases outwards.

Table 8.1: Table of ZAMS parameters
(from Shore "The Tapestry of Modern Astrophysics")

$\frac{M}{M_\odot}$	$\log\frac{L}{L_\odot}$	$\log\frac{T_{\rm eff}}{\rm K}$	$\frac{T_c}{10^7\rm K}$	$\frac{R}{R_\odot}$
0.3	−1.90	3.55	0.8	0.3
0.5	−1.42	3.59	0.9	0.44
1.0	−0.15*	3.75	1.4	0.9*
2.0	1.22	3.97	2.2	1.6
3.0	1.91	4.09	2.4	2.0
5.0	2.74	4.24	2.8	2.6
9.0	3.61	4.38	3.1	3.6
15.0	4.32	4.51	3.4	4.7

* our Sun is already chemically evolved

- if we assume $H_p = const.$ throughout star we get (in linear approximation)

$$P = P_0 e^{-r/H_p}$$

$H_p =$ distance over which gas pressure decreases by factor of e.

- hydrostatic equilibrium (pressure grad. balances gravity):

$$\frac{dP}{dr} = -g\varrho = -\frac{GM(r)}{r^2}\varrho$$

$$H_p = \frac{P}{g\varrho} = \frac{Pr^2}{GM(r)\varrho}$$

- **Note:** a typical value for the Sun is $H_p \approx 1/10\,\mathrm{R}_\odot$!

8.5 Adiabatic Temperature Gradient

- Start with ideal gas law: ($PV = NkT$)

$$P_{gas} = \frac{\varrho kT}{\mu m_p}$$

$$\frac{dP_{gas}}{dr} = -\frac{P}{\mu}\frac{d\mu}{dr} + \frac{P}{\varrho}\frac{d\varrho}{dr} + \frac{P}{T}\frac{dT}{dr}$$

$\mu =$ mean mol. weight
$V = \varrho^{-1}$ specific volume

- Use polytropic EOS: $P = K\varrho^{\gamma}$

$$\frac{dP}{dr} = \gamma \frac{P}{\varrho}\frac{d\varrho}{dr}$$

- Let´s assume now $\mu =$ const., then we can combine both equations.

$$\gamma \frac{P}{\varrho}\frac{d\varrho}{dr} = \frac{P}{\varrho}\frac{d\varrho}{dr} + \frac{P}{T}\frac{dT}{dr}$$

$$\left.\frac{dT}{dr}\right|_{ad} = (\gamma - 1)\frac{T}{\varrho}\frac{d\varrho}{dr}$$

$$\text{OR:} \quad \left.\frac{dT}{dr}\right|_{ad} = \left(1 - \frac{1}{\gamma}\right)\frac{I}{P}\frac{dP}{dr} \tag{8.3}$$

in hydrostatic balance, we can express $\frac{dP}{dr}$ by the gravitational force:
$\frac{dP}{dr} = -GM(r)r^{-2}\varrho$

$$\left.\frac{dT}{dr}\right|_{ad} = -\left(1 - \frac{1}{\gamma}\right)\frac{T}{P}\frac{GM(r)}{r^2}\varrho$$

with ideal gas law: $P = \dfrac{\varrho kT}{m_p \mu}$ we finally get

$$\left.\frac{dT}{dr}\right|_{ad} = -\left(1 - \frac{1}{\gamma}\right)\frac{\mu m_p}{k}\frac{GM(r)}{r^2} \qquad \textbf{adiabatic temperature gradient}$$

- rephrase using: $g = GM(r)r^{-2}$ and $nR = k(\mu m_p)^{-1}$ and $\gamma = c_p/c_v$ and $c_p - c_v = nR$

$$\left.\frac{dT}{dr}\right|_{ad} = -\frac{g}{c_p}$$

- think of a hot bubble that **rises** due to **buoyancy effects:** if it rises fast enough such heat transfer with surrounding is negligible, it evolves **adiabatically.**
  it is in **pressure equilibrium** with its surrounding (if sound-waves have sufficient time to travel across, which is usually the case).
  after some time, finally thermalizes with surrounding and losses its identity and dissolves.

- the temperature evolution of bubble is given by

$$\left.\frac{dT}{dr}\right|_{ad} = -\frac{g}{c_p}$$

- if the actual temperature gradient in the star is **LARGER** than the adiabatic value, i.e. if it is **SUPERADIABATIC**, then **CONVECTION** sets in.

  buoyancy effects grow as the bubble climbs up and the upwards motion continues $\hookrightarrow$ convective instability

- **Criterium for convection:**

$$\left|\frac{dT}{dr}\right|_{act} > \left|\frac{dT}{dr}\right|_{ad}$$

  convection sets in even if $\left|\frac{dT}{dr}\right|_{act}$ is only slightly larger than $\left|\frac{dT}{dr}\right|_{ad}$. this sufficient to carry almost all energy outwards by convection.

- if convection sets in, it brings the **effective** temperature gradient close to the convective/adiabatic value.

$$\left|\frac{dT}{dr}\right|_{\text{eff}} \approx \left|\frac{dT}{dr}\right|_{ad}$$

- **fully convective stars** can be well described as **polytropes** with $\gamma = 5/3$, i.e. with index $\mathbf{n} = (\gamma - \mathbf{1})^{-\mathbf{1}} = \mathbf{3/2}$.

- we can find another criterion for the onset of convection:
  - Revisit equation (8.3)

$$\left.\frac{dT}{dr}\right|_{ad} = \left(1 - \frac{1}{\gamma}\right)\frac{T}{P}\frac{dP}{dr}$$

  - and recall $\left|\frac{dT}{dr}\right|_{act} > \left|\frac{dT}{dr}\right|_{ad}$ for instability

$$\text{combine:} \quad \underbrace{\left|\frac{dT}{dr}\right|_{act}}_{<0} > \left|\underbrace{\left(1 - \frac{1}{\gamma}\right)}_{>0}\frac{T}{P}\underbrace{\frac{dP}{dr}}_{<0}\right|$$

  - both T and P decrease with increasing radius r
  - $\left(1 - \frac{1}{\gamma}\right) > 0$ as $\gamma < 1$

$$\begin{aligned}
\left.\frac{dT}{dr}\right|_{act} &> \left(1 - \frac{1}{\gamma}\right)\frac{T}{P}\frac{dP}{dr} \\
\frac{P}{T}\frac{dT}{dP} &> 1 - \frac{1}{\gamma} \\
\frac{d\ln T}{d\ln P} &> \frac{\gamma - 1}{\gamma} \longrightarrow \frac{d\ln P}{d\ln T} < \frac{\gamma}{\gamma - 1}
\end{aligned}$$

- for an ideal monoatomic gas, **convection** will assure in regions of the star where

$$\frac{d\ln p}{d\ln T} < 2.5 \qquad \text{(recall } \gamma = 5/3\text{)}$$

in this case the temperature gradient is

$$\frac{dT}{dr} \approx \left.\frac{dT}{dr}\right|_{ad} = -\left(1 - \frac{1}{\gamma}\right)\frac{\mu m_p}{k}\frac{GM_r}{r} \tag{8.4}$$

Otherwise, **radiation dominates** and

$$\frac{dT}{dr} = -\frac{3}{4}\frac{\langle\kappa\rangle\varrho}{acT^3}\frac{L_r}{4\pi r^2} \tag{8.5}$$

- another way of looking at the question of when convection dominates over radiation comes from comparing (8.4) with (8.5):

$$-\left(1 - \frac{1}{\gamma}\right)\frac{\mu m_p}{k}\frac{GM_r}{r} \approx -\frac{3}{4ac}\frac{\langle\kappa\rangle\varrho}{T^3}\frac{L_r}{4\pi r^2}$$

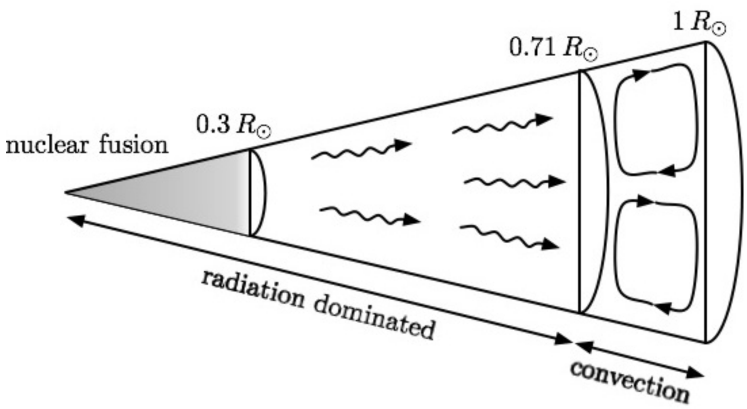
Convection occurs:

(1) when the **stellar opacity** is **large**, then an unachievable temperature gradient would be necessary for transporting away all energy by radiation
$\hookrightarrow$ **convection then is more efficient for energy transport**

(2) in regions where **ionization** occurs, because of **large specific heat** and low adiabatic gradients.

(3) when the **temperature dependence** of nuclear energy generation is **large**, causing steep radiative flux gradients and thus a large temperature difference

In stellar atmospheres often (1) + (2) are met simultaneously (e.g. Sun)



9 Nuclear Energy Source

- For nuclei less massive than iron, nuclear fusion releases energy.
  For heavier elements, energy is released by “fission”.

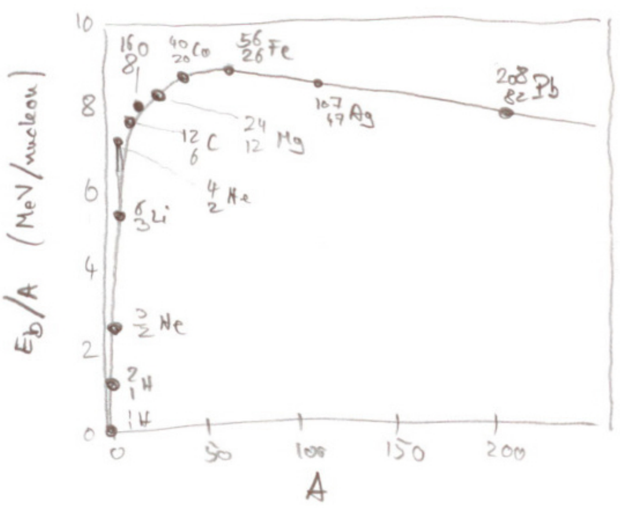


Figure 9.1:

- **Example:** Can nuclear fusion power Sun for long enough?

  Assume:

  - initially 100% hydrogen
  - only 10% of Sun can burn H to He
  - 0.7% of H mass gets converted into energy when forming He.

$$
\begin{aligned}
E_{nucl} &\approx 0.1 \cdot 0.007 \, \mathrm{M}_\odot \, \mathrm{c}^2 \\
&\approx 1.3 \cdot 10^{51} \, \mathrm{erg}
\end{aligned}
$$

time scale:

$$
\begin{aligned}
t_{nucl} &\approx \frac{E_{nucl}}{L_\odot} \approx \frac{1.3 \cdot 10^{51}\,\mathrm{erg}}{4 \cdot 10^{33}\,\mathrm{erg}}\mathrm{s} \\
&\approx 3.25 \cdot 10^{17}\,\mathrm{s} \\
&\approx 10^{10}\,\mathrm{yr}
\end{aligned}
$$

compare to gravitational contraction timescale:

$$t_{\text{Kelvin-Helmholtz}} \approx 30 \cdot 10^{6}\,\mathrm{yr}$$

$$\mathbf{t_{nucl} \gg t_{KH}}$$

- Quantum mechanical tunneling is necessary for nuclear fusion to work in the stellar interior.

  To show that, check temperature required in the fully classical approximation.

  we ask: what temperature is required for two protons to come close enough so that strong forces overwhelm Coulomb repulsion?

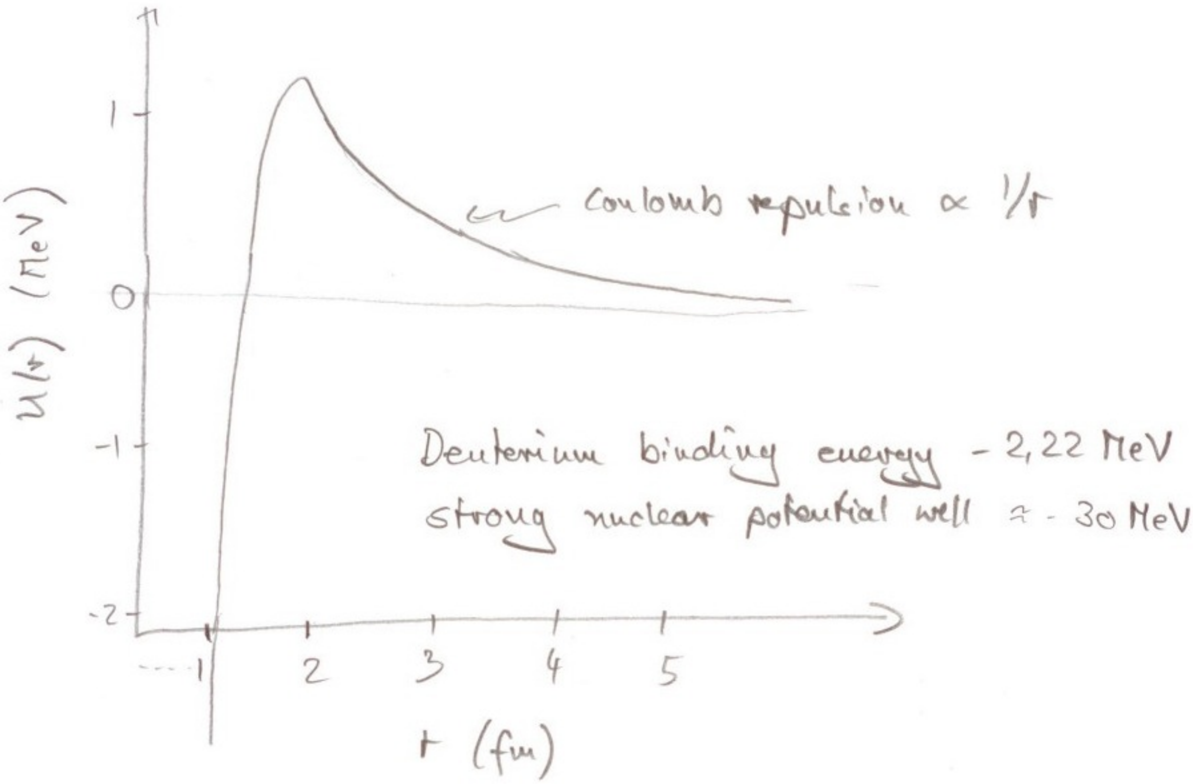


Figure 9.2: p-p interaction

use Maxwell-Boltzmann velocity distribution.

peak:

$$\frac{1}{2}\mu_m\langle v^2\rangle = \frac{3}{2}kT = \frac{1}{4\pi\epsilon_0}\frac{Z_1Z_2e^2}{r}$$

μ_m = reduced mass

$$T = \frac{1}{6\pi\epsilon_0}\frac{Z_1Z_2e^2}{kr}$$

for $Z_1 = Z_2 = 1$ and $r \approx 1\,\mathrm{fm} = 10^{-15}\,\mathrm{m}$

$$\Rightarrow \quad T \approx 10^{10}\,\mathrm{K}$$

this is too hot compared to $T_c = 14 \cdot 10^6\,\mathrm{K}$ of the Sun.
even in Maxwell-tail is considered (instead of peak)

Other process required for fusion ⇒ **tunneling**

Approximation: tunneling becomes important when De Broglie wavelength of particle gets of order of the extent of potential well, i.e. of order fm.

kinetic energy:

$$\frac{1}{2}\mu_m\langle v^2\rangle = \frac{p^2}{2\mu_m} = \frac{(h/\lambda)^2}{2\mu_m} \qquad \text{(Heisenberg: } \lambda p_\lambda \approx h)$$

put together:

$$\frac{1}{4\pi\epsilon_0}\frac{Z_1Z_2e^2}{\lambda} = \frac{(h/\lambda)^2}{2\mu_m}$$

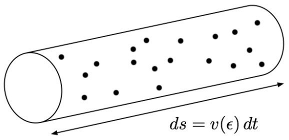
$$T = \frac{\mu_m}{12\pi\epsilon_0^2}\frac{Z_1^2Z_2^2e^4}{h^2k}$$

Again for collision of two protons: $\mu_m = \frac{1}{2}m_p$ and $Z_1 = Z_2 = 1$

$$\Rightarrow \quad T \approx 10^7\,\mathrm{K}$$

this temperature is consistent with estimates of stellar interior

9.1 Nuclear reaction rate and Gamow peak



number of reactions per nucleus per time interval = probability for reaction X incident flux

- $dN_E = \sigma(E)v(E)n_i\, dE\, dt$

$$\begin{array}{lcl} \sigma(E) & = & \text{cross section} \\ v(E) & = & \text{velocity, corresponding to } E \\ n_i & = & \text{number of incident particles with energy } E \end{array}$$

- $v(E) = \sqrt{\dfrac{2E}{\mu_m}}$

- number of reactions per second per nucleus

$$\frac{dN_E}{dt} = \sigma(E)v(E)\frac{n_i}{n}n_E\, dE$$

$$\begin{array}{lcl} n_E\, dE & = & 2n\sqrt{\dfrac{E}{\pi(kT)^3}}e^{-E/kT}\, dE \\ & = & \text{number of particles with } E \text{ in } [E,\, E+dE] \\ & = & \text{from Maxwell-Boltzmann distribution} \end{array}$$

- reaction-rate between incident flux $n_i v(E)$ and n_x targets is then

$$R_{ix} = \int_0^\infty n_x n_i \sigma(E)v(E)\frac{n_E}{n}\, dE$$

- now obtain $\sigma(E)$: with **tunneling probability** we get:

$$\sigma(E) \propto e^{-2\pi U_c/E} \tag{9.1}$$

  where U_c = height of Coulomb barrier.

  because

$$\frac{U_c}{E} = \frac{Z_1 Z_2 e^2}{2\pi\epsilon_0 hv} \propto E^{-1/2}$$

  we get:

$$\sigma(E) \propto e^{-bE^{1/2}}$$

with $b = \sqrt{\frac{\mu_m}{2}} \frac{\pi Z_1 Z_2 e^2}{\epsilon_0 h}$

Also from estimate of De Broglie wavelength:

$$\sigma(E) \propto \pi\lambda^2 \propto \pi \left(\frac{n}{p}\right)^2 \propto \frac{1}{E} \tag{9.2}$$

Both "together":

$$\sigma(E) = \frac{S(E)}{E} e^{-bE^{-1/2}}$$

with $S(E)$ slowly varying in E.

Putting all together:

$$R_{ix} = \left(\frac{2}{kT}\right)^{3/2} \frac{n_i n_x}{\sqrt{\mu_m \pi}} \int_0^\infty S(E) e^{-bE^{-1/2}} e^{-E/kT}\, dE$$

numbers:

$$R_{ix} = 6.48 \cdot 10^{-24} \frac{n_i n_x}{\mu_m Z_1 Z_2} S(E_0) \left(\frac{E_G}{4kT}\right)^{2/3} \exp\left(-3\left[\frac{E_G}{4kT}\right]^{1/3}\right) \frac{1}{m^3 s}$$

$$R_{ix} \propto n_i n_x \left(\frac{E_G}{4kT}\right)^{2/3} \exp\left(-3\left[\frac{E_G}{4kT}\right]^{1/3}\right)$$

$e^{-E/kT}$	:	high energy wing of MB distribution
$e^{-bE^{-1/2}}$	:	comes from penetration probability

together **strongly peaked curve!**

Maximum at $E_0 = \left(\frac{bkT}{2}\right)^{2/3}$

besides this continuum description, there may be **Resonances** and **Electron screening** effects (polarization).

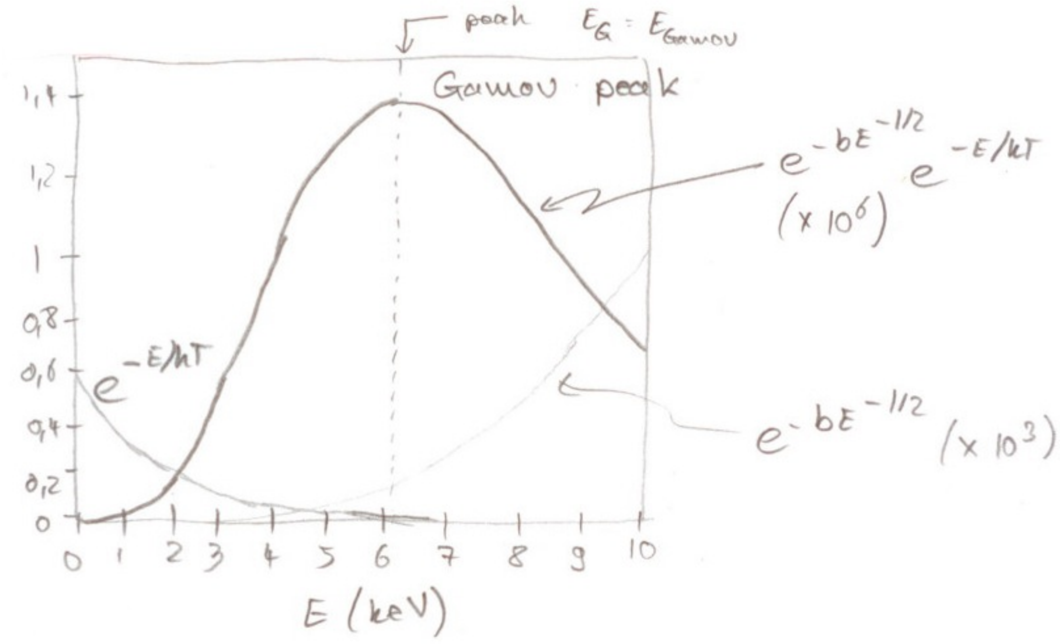


Figure 9.3:

9.2 Power-law description

- often the rate R_{ix} can be described as power law:

  $$R_{ix} \approx R_0 \, X_0 \, X_x \, \varrho^{\alpha'} \, T^{\beta}$$

  if energy per reaction is ϵ_{0x}, then the total energy per unit mass is

  $$\epsilon_{ix} = \frac{\epsilon_{0x}}{\varrho} R_{ix} = \epsilon_{0x} \, X_i \, X_x \, \varrho^{\alpha} \, T^{\beta} \qquad (\alpha = \alpha' - 1)$$

- total energy released, sum over all reactions

  $$\epsilon = \sum_x \epsilon_{0x} \, X_i \, X_x \, \varrho^{\alpha} \, T^{\beta}$$

- the resulting luminosity gradient:

  $$dL = \epsilon \, dm$$

  where $\epsilon = \epsilon_{nuclear} + \epsilon_{gravity}$

  Note: $\epsilon_{gravity}$ can be negative if star is expanding!

- with $dm = dM_r = \varrho \, dV = 4\pi r^2 \varrho \, dr$

  $$\frac{dL_r}{dr} = 4\pi r^2 \varrho \epsilon$$

- this is the last stellar structure equation.
  now we have everything to compute the structure of star as function of mass, radius, composition, etc.

9.3 Nuclear Reaction Rates

- Stars consists mostly of protons, so let´s starts with

  $$\mathrm{p} + \mathrm{p} \longrightarrow \mathrm{d} + \mathrm{e}^+ + \nu_\mathrm{e}$$

  this would be fast if it were not to need a **weak** interaction (to convert $\mathrm{p} \rightarrow \mathrm{n}$)

  $\mathrm{p} \rightarrow \mathrm{n} + \mathrm{e}^+ + \nu$ needs 1.8 MeV, but this is paid back by binding energy of deuteron, which is 2.22 MeV

  the p-p fusion rate is $5 \cdot 10^{13}\,\mathrm{m}^{-3}\mathrm{s}^{-1}$

  - a proton in center of Sun hangs around for $9 \cdot 10^9\,\mathrm{yr}$
    on average before it fuses with other proton
  - this sets **nuclear timescale**.

- once there is d, it quickly reacts with another p to build $^3\mathrm{He}$.

  $$\mathrm{p} + \mathrm{d} \longrightarrow {}^3\mathrm{He} + \gamma$$

- now there are several ways to make $^4\mathrm{He}$.

  I) ${}^3_2\mathrm{He}$ fuses with ${}^3_2\mathrm{He}$ to make ${}^4_2\mathrm{He} + 2\mathrm{p}$

  $${}^3_2\mathrm{He} + {}^3_2\mathrm{He} \longrightarrow {}^4_2\mathrm{He} + 2\mathrm{p}$$

  $$\Delta E_\mathrm{eff} = 26.2\,\mathrm{MeV}$$

  II) or ${}^3_2\mathrm{He}$ fuses with ${}^4_2\mathrm{He}$ to ${}^7_4\mathrm{Be}$ and

  $$\begin{aligned} {}^3_2\mathrm{He} + {}^4_2\mathrm{He} &\longrightarrow {}^7_4\mathrm{Be} + \gamma \\ {}^7_4\mathrm{Be} + \mathrm{e}^- &\longrightarrow {}^7_3\mathrm{Li} + \nu_\mathrm{e} \\ {}^7_3\mathrm{Li} + \mathrm{p} &\longrightarrow 2\,{}^4_2\mathrm{He} \end{aligned}$$

  $$\Delta E_\mathrm{eff} = 25.2\,\mathrm{MeV}$$

III) or ${}^{3}_{2}\mathrm{He}$ fuses again with ${}^{4}_{2}\mathrm{He}$ to ${}^{7}_{4}\mathrm{Be}$, but now

$$
\begin{aligned}
{}^{3}_{2}\mathrm{He} + {}^{4}_{2}\mathrm{He} &\longrightarrow {}^{7}_{4}\mathrm{Be} + \gamma \\
{}^{7}_{4}\mathrm{Be} + \mathrm{p} &\longrightarrow {}^{8}_{5}\mathrm{B} + \gamma \\
{}^{8}_{5}\mathrm{B} &\longrightarrow {}^{8}_{4}\mathrm{Be} + \mathrm{e}^{+} + \nu \\
{}^{8}_{4}\mathrm{Be} &\longrightarrow 2\,{}^{4}_{2}\mathrm{He}
\end{aligned}
$$

$$\Delta E_{\mathrm{eff}} = 19.1\,\mathrm{MeV}$$

- The **rate limiting reaction** is the very first one:

$$
\begin{aligned}
S_{\mathrm{pp}}(0) &= 3.8 \cdot 10^{-22}\,\mathrm{keV\,barn} \\
S_{\mathrm{pd}}(0) &= 2.5 \cdot 10^{-4}\,\mathrm{keV\,barn}
\end{aligned}
$$

this is 10^{18} times larger! (barn $= 10^{-24}\,\mathrm{cm}^2$)

- in summary: **pp Chain** (dominant for solar-type stars)

$$
\begin{aligned}
{}^{1}_{1}\mathrm{H} + {}^{1}_{1}\mathrm{H} &\longrightarrow {}^{2}_{1}\mathrm{H} + \mathrm{e}^{+} + \nu_{\mathrm{e}} \\
{}^{2}_{1}\mathrm{H} + {}^{1}_{1}\mathrm{H} &\longrightarrow {}^{3}_{2}\mathrm{H} + \gamma
\end{aligned}
$$

Note: branching ratios differ from Carroll and Ostlie [2006] to Phillips

69% [85%]

$${}^{3}_{2}\mathrm{He} + {}^{3}_{2}\mathrm{He} \longrightarrow {}^{4}_{2}\mathrm{He} + 2\,{}^{1}_{1}\mathrm{H}$$

ppI

31% [15%]

$${}^{3}_{2}\mathrm{He} + {}^{4}_{2}\mathrm{He} \longrightarrow {}^{7}_{4}\mathrm{Be} + \gamma$$

99.7%

$$
\begin{aligned}
{}^{7}_{4}\mathrm{Be} + \mathrm{e}^{-} &\longrightarrow {}^{7}_{3}\mathrm{Li} + \nu_{\mathrm{e}} \\
{}^{7}_{3}\mathrm{Li} + {}^{1}_{1}\mathrm{H} &\longrightarrow 2\,{}^{4}_{2}\mathrm{He}
\end{aligned}
$$

ppII

0.3%

$$
\begin{aligned}
{}^{7}_{4}\mathrm{Be} + {}^{1}_{1}\mathrm{H} &\longrightarrow {}^{8}_{5}\mathrm{B} + \gamma \\
{}^{8}_{5}\mathrm{Be} &\longrightarrow {}^{8}_{4}\mathrm{Be} + \mathrm{e}^{+} + \nu_{\mathrm{e}} \\
{}^{8}_{4}\mathrm{Be} &\longrightarrow 2\,{}^{4}_{2}\mathrm{He}
\end{aligned}
$$

ppIII

effective energy gain:

$$\begin{aligned}\Delta E_{\text{eff}} &= 69\% \cdot \frac{26.2\,\text{MeV}}{2} + 31\% \cdot 25.2\,\text{MeV} \\ &= 16.9\,\text{MeV} \qquad \text{[Carroll and Ostlie, 2006]}\end{aligned}$$

$$\begin{aligned}\Delta E_{\text{eff}} &= 0.85 \cdot \frac{26.2\,\text{MeV}}{2} + 0.15 \cdot 25.2\,\text{MeV} \\ &= 15\,\text{MeV} \qquad \text{(Phillips)}\end{aligned}$$

- all together:

$$\epsilon_{pp} \approx 1.08 \cdot 10^{-5}\,\frac{\text{erg/s}}{\text{cm}^3\text{g}}\,\varrho X^2 T^4 \underbrace{f_{pp}\phi_{pp}C_{pp}}_{QM factors \approx 1} \qquad ; X = H \text{ fraction}$$

$$\epsilon_{pp} \propto T_6^4 \quad \text{at} \quad T_6 \approx 15 \qquad (**) T_6 = 10^6\,\text{K}$$

this is **very modest** T dependency.

- **CNO cycle for massive star´s**
  - while the pp cycle can account for hydrogen burning in solar-type main sequence star´s it fails for massive star´s.
  - as $L \propto M^{5.5}$ (for stars similar to Sun) even a modest mass increase results in enormous luminosity gain. this is too much for the “modest” T^4-dependency of pp chain
    (recall: $T \propto M/R$)
    $\hookrightarrow$ something else must govern heat production in massive star´s.
  - steeper T-dependency required → larger Coulomb barrier → must involve heavy elements → but as their abundance (at best!) is very low, they must be recycled to prolong H burning
  - **CNO cycle with carbon, nitrogen, oxygen as catalysts!**

$$\mathrm{p} + {}^{12}_{6}\mathrm{C} \longrightarrow \underbrace{{}^{13}_{7}\mathrm{N}}_{{}^{13}_{6}\mathrm{C}+\mathrm{e}^+ + \nu_e} + \gamma \qquad S(0) = 1.5\,\text{keV barn}$$

$$\mathrm{p} + {}^{13}_{6}\mathrm{C} \longrightarrow {}^{14}_{7}\mathrm{N} + \gamma \qquad S(0) = 5.5\,\text{keV barn}$$

$$\mathrm{p} + {}^{14}_{7}\mathrm{N} \longrightarrow \underbrace{{}^{15}_{8}\mathrm{O}}_{{}^{15}_{7}\mathrm{N}+\mathrm{e}^+ + \nu_e} + \gamma \qquad S(0) = 3.3\,\text{keV barn}$$

$$\mathrm{p} + {}^{15}_{7}\mathrm{N} \longrightarrow {}^{12}_{6}\mathrm{C} + {}^{4}_{2}\mathrm{He} \qquad S(0) = 78\,\text{keV barn}$$

the net result is

$$4\mathrm{p} \longrightarrow {}^{4}_{2}\mathrm{He} + 3\gamma + 2\mathrm{e}^+ + 2\mathrm{e}\nu \qquad \Delta E_{\text{eff}} = 23.8\,\text{MeV}$$

– just like in pp, there are several CNO chains

$$
\begin{aligned}
{}^{12}_{6}\mathrm{C} + {}^{1}_{1}\mathrm{H} &\longrightarrow {}^{13}_{7}\mathrm{N} + \gamma \\
{}^{13}_{7}\mathrm{N} &\longrightarrow {}^{13}_{6}\mathrm{C} + \mathrm{e}^{+} + \nu_{\mathrm{e}} \\
{}^{13}_{6}\mathrm{C} + {}^{1}_{1}\mathrm{H} &\longrightarrow {}^{14}_{7}\mathrm{N} + \gamma \\
{}^{14}_{7}\mathrm{N} + {}^{1}_{1}\mathrm{H} &\longrightarrow {}^{15}_{8}\mathrm{O} + \gamma \\
{}^{15}_{8}\mathrm{O} &\longrightarrow {}^{15}_{7}\mathrm{N} + \mathrm{e}^{+} + \nu
\end{aligned}
$$

99.96%

$$
{}^{15}_{7}\mathrm{N} + {}^{1}_{1} \longrightarrow {}^{12}_{6}\mathrm{C} + {}^{4}_{2}\mathrm{He}
$$

0.04%

$$
\begin{aligned}
{}^{15}_{7}\mathrm{N} + {}^{1}_{1}\mathrm{H} &\longrightarrow {}^{16}_{8}\mathrm{O} + \gamma \\
{}^{16}_{8}\mathrm{O} + {}^{1}_{1}\mathrm{H} &\longrightarrow {}^{17}_{8}\mathrm{F} + \gamma \\
{}^{17}_{8}\mathrm{F} &\longrightarrow {}^{17}_{8}\mathrm{O} + \mathrm{e}^{+} + \nu_{\mathrm{e}} \\
{}^{17}_{8}\mathrm{O} + {}^{1}_{1}\mathrm{H} &\longrightarrow {}^{14}_{7}\mathrm{N} + {}^{4}_{2}\mathrm{He}
\end{aligned}
$$

– $\epsilon_{CNO} = 8.24 \cdot 10^{-24} \dfrac{\mathrm{erg/s}}{\mathrm{cm}^3\,\mathrm{g}^2}\, \varrho\, X Z\, T_6^{19.9} \qquad \text{at} \quad T_6 \approx 15$

very steep T dependency:

$$
\epsilon_{\mathrm{CNO}} \propto T^{20} \qquad \text{at} \quad T_6 \approx 15
$$

X = H fraction

Z = “metal” fraction

- **Comparison** pp **& CNO:**
  - for Sun ($T_6 = 15,\ X = 0.7,\ Z \approx 0.02$)

    $$\epsilon_{\rm pp} \approx 10 \cdot \epsilon_{\rm CNO}$$

    only 10% of Sun´s energy production from CNO [Note: Phillips (**) gives 2%]

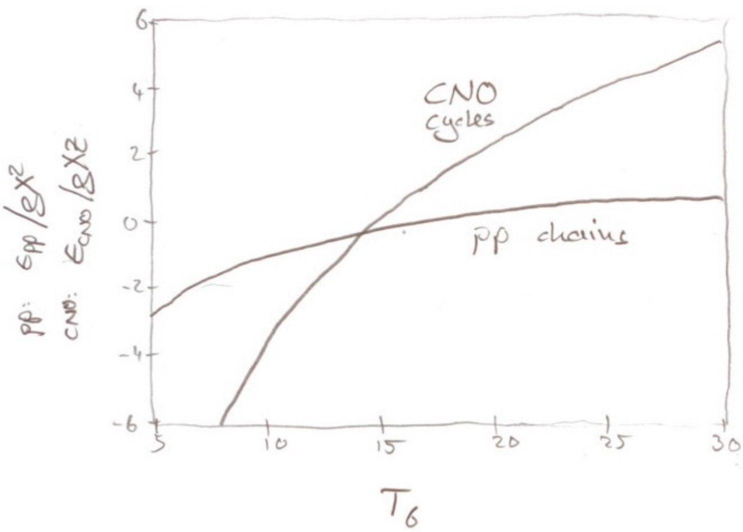


Figure 9.4:

  - pp chains are lower in efficiency and energy yield than CNO (once CNO becomes possible)

    $$\left.\begin{aligned} \epsilon_{\rm pp} &\approx 11\ldots17\,{\rm MeV} \\ \epsilon_{\rm CNO} &\approx 23\ldots27\,{\rm MeV} \end{aligned}\right\} \quad \text{for solar models}$$

- **Triple α-Process of Helium Burning**

  once the central density and temperature gets high enough, He burning can set in:

  $$\begin{aligned} {}^4_2{\rm He} + {}^4_2{\rm He} &\longleftrightarrow {}^8_4{\rm Be}^* \\ {}^8_4{\rm Be} + {}^4_2{\rm He} &\longrightarrow {}^{12}_6{\rm C} + \gamma \end{aligned}$$

  $$\Delta E_{\rm eff} = 7.3\,{\rm MeV}$$

  * unstable, decays back to $2\,{}^4_2$He if not hit by other ${}^4_2$He

  $$\epsilon_{3\alpha} = \epsilon_{0.3\alpha}\,\varrho^2\,Y^3\,T_8^{41} \quad (Y = \text{He fraction})$$

  very steep T dependence: $\Delta T = 10\% \rightarrow \Delta L$ of 5000%!

- **Carbon and Oxygen Burning**

$$\begin{aligned} {}^{12}_{6}\mathrm{C} + {}^{4}_{2}\mathrm{He} &\longrightarrow {}^{16}_{8}\mathrm{O} + \gamma \\ {}^{16}_{8}\mathrm{O} + {}^{4}_{2}\mathrm{He} &\longrightarrow {}^{20}_{10}\mathrm{Ne} + \gamma \end{aligned}$$

  Production of α-Elements!

  Near $6 \cdot 10^8$K:

$${}^{12}_{6}\mathrm{C} + {}^{12}_{6}\mathrm{C} \longrightarrow \begin{cases} {}^{16}_{8}\mathrm{O} + 2\,{}^{4}_{2}\mathrm{He} & \Delta E < 0 \\ {}^{20}_{10}\mathrm{Ne} + {}^{4}_{2}\mathrm{He} & \Delta E > 0 \\ {}^{23}_{11}\mathrm{No} + \mathrm{p}^{+} & \Delta E > 0 \\ {}^{23}_{11}\mathrm{Mg} + \mathrm{n} & \Delta E < 0 \\ {}^{24}_{12}\mathrm{Mg} + \gamma & \Delta E > 0 \end{cases}$$

  Further oxygen burning at $T > 10^9\,\mathrm{K}$

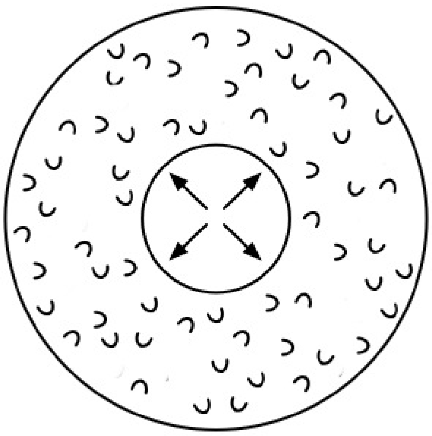
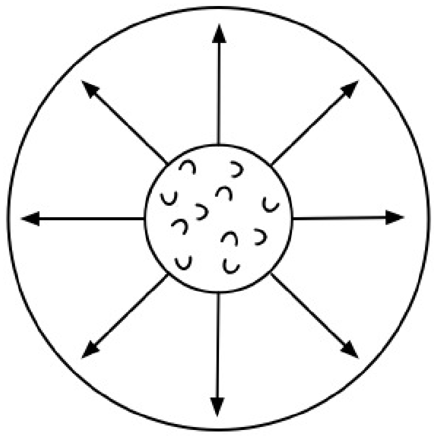
$${}^{16}_{8}\mathrm{O} + {}^{16}_{8}\mathrm{O} \longrightarrow \begin{cases} {}^{24}_{12}\mathrm{Mg} + 2\,{}^{4}_{2}\mathrm{He} & \Delta E < 0 \\ {}^{28}_{14}\mathrm{Si} + {}^{4}_{2}\mathrm{He} & \Delta E > 0 \\ {}^{31}_{15}\mathrm{P} + \mathrm{p}^{+} & \Delta E > 0 \\ {}^{31}_{16}\mathrm{S} + \mathrm{n} & \Delta E > 0 \\ {}^{32}_{16}\mathrm{S} + \gamma & \Delta E > 0 \end{cases}$$

  $\Delta E < 0$: energy is absorbed rather than released.

- Fusion goes up to ${}^{56}\mathrm{Fe}$!

  This is the most stable element.

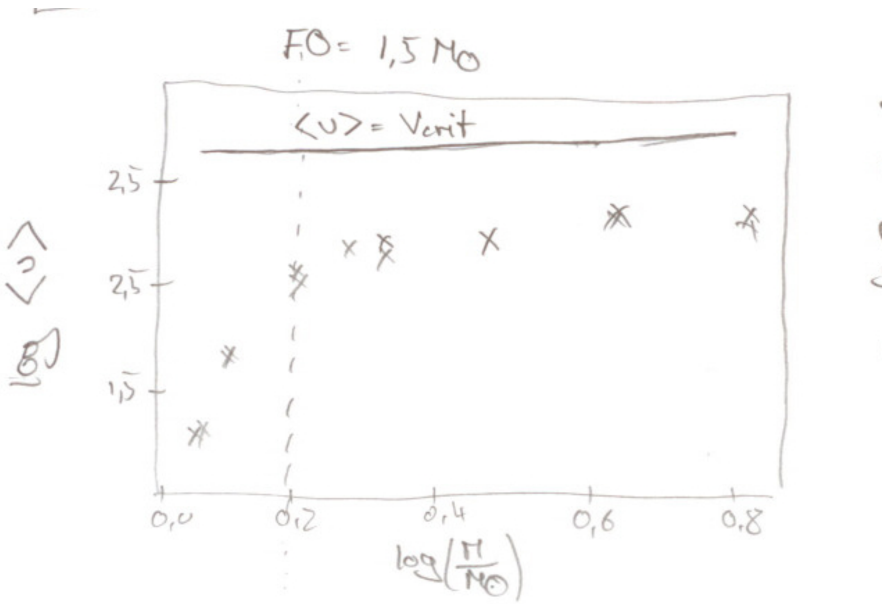
9.4 Summary of Structure

lower main sequence	upper main sequence
$M < 1.5\,\mathrm{M}_\odot \quad (FO)$	$M > 1.5\,\mathrm{M}_\odot \quad (FO)$
pp-chain	CNO-cycle
$\epsilon_{\mathrm{pp}} \propto T^4$	$\epsilon_{\mathrm{CNO}} \propto T^{20}$
low T-dependency; less concentrated energy source $\rightarrow$ small T-gradients	in center larger flux and steep temperature gradient
Radiative Core	**Convective Core**
$T_c \quad >$	$20 \cdot 10^6 \quad > \quad T_c$
surface H neutral ; then ionization and rapid increase of opacity $\rightarrow$ steep T gradient.	surface hot and ionized modest temperature gradient.
convective envelope	**radiative envelope**
	
the smaller M, the further convective zone reaches into star.	the larger m, the larger the convective core

10 Effects of Rotation

- **Don Clayton:** Rotation effects the evolution of a star in at least two ways:
  (1) Conservation of angular momentum during structural changes (contraction /expansion)
  (2) onset of fluid circulation to maintain energy balance.

  (1) centrifugal forces render rotating star **non-spherical!**
  (2) **von Zeipels Theorem & gravity darkening**
- Let´s start with observation:
  - massive stars rotate rapidly
  - low-mass stars rotate slowly

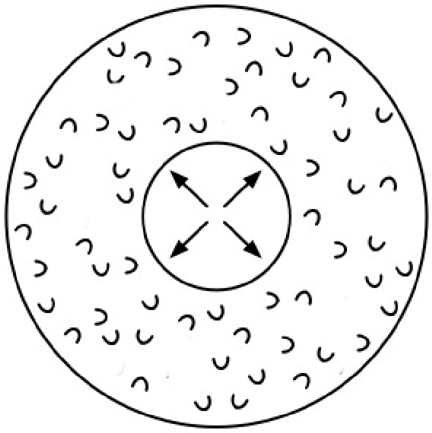


$$V_{\mathrm{circ}} = \text{break-up velocity} = \sqrt{\frac{GM}{R}}$$

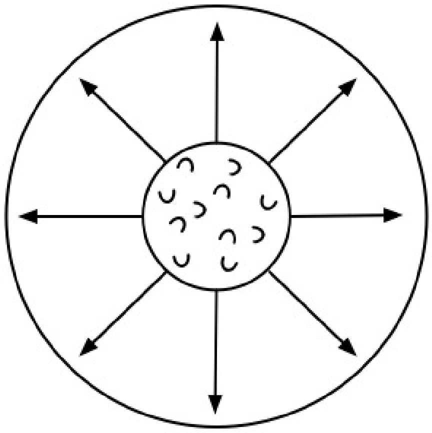
Figure 10.1: Plot of average equatorial velocity for different stellar populations.

– Stars with masses below $1.5\,M_\odot$ (later than FO) have significant convection in envelope

Recall:



lower main sequence: core radiation envelope convective



upper main sequence: core convective envelope radiative

– Winds can carry away **angular momentum:**

The amount of angular momentum loss depends at radius at which wind decouples from stellar interior:

– Simple winds decouples near the photosphere.

But: if wind couples to magnetic field it can be forced to **corotate** with the star well **beyond** the photosphere!

The B-field is rooted in stellar interior and as corotating wind moves beyond the photosphere it gains angular momentum at the expense of the interior.

– low-mass star´s have significant convection

$\hookrightarrow$ relatively strong fields & winds.
$\hookrightarrow$ strong angular momentum loss
high-mass star´s weaker fields $\rightarrow$ weaker mass loss.

- Implication:

  Stars slow down during their evolution!

- **von Zeipel´s theorem**
  - consider a star that is in mechanical & thermal equilibrium
    - rotation alters the shape of star
    - equations of hydrostatic equilibrium must include **centrifugal force**
  - centrifugal acceleration can be associated with **centrifugal potential:**

    Force: $\Omega^2 R\,\vec{e}_R = -\vec{\nabla}\psi$

    $$\psi = -\int_0^R \Omega^2 R'\,dR' = -\frac{1}{2}\Omega^2 R^2$$

    for rigid rotator.
  - effective potential:

    $$\phi_{\text{eff}} = \phi + \psi = \phi - \frac{\Omega^2 R^2}{2}$$

  - equation of hydrostatic equilibrium:

    $$\begin{aligned} \frac{1}{\varrho}\vec{\nabla}P &= -\vec{\nabla}\phi_{\text{eff}} = -\vec{\nabla}\phi + \Omega^2\,\vec{e}_R \\ &= \vec{g}_{\text{eff}} = \text{acceleration due to local effective gravity.} \end{aligned} \tag{10.1}$$

    also: Poisson: $\vec{\nabla}^2\phi = 4\pi G\varrho$ (note, culy true gravity, ϕ, appears here.)

- consider "level" surfaces of **constant** $\boldsymbol{\phi_{\text{eff}}}$

  $\vec{\nabla}\phi_{\text{eff}}$ evaluated on such surface is always 1 to surface

  consider $d\phi_{\text{eff}} = d\vec{r}\cdot\vec{\nabla}\phi_{\text{eff}}$, with $d\vec{r}$ being tangent vector

  $d\phi_{\text{eff}} = 0$

  back in (10.1) $\rightarrow$ $\varrho^{-1}d\vec{r}\cdot\vec{\nabla}P = 0$ as well

  $P =$ constant an equipotential surface

  for barotropic (10.1): $P = P(\varrho)$

  $P = P(\phi_{\text{eff}})$ & $\phi_{\text{eff}} = \phi_{\text{eff}}(P)$

  with (10.1) $\varrho^{-1} = \dfrac{d\phi_{\text{eff}}}{dP}$ = constant on equipotential surface

  $\varrho = \varrho(\phi_{\text{eff}})$

- now consider chemically homogeneous star (μ =const.)

$$P = \frac{\varrho k T}{\mu m_p} \Rightarrow T = T(\phi_{\text{eff}})$$

- So, the only "thing" not constant on equipol. surface is effective acceleration $\vec{g}_{\text{eff}}$. it is always $\perp$ to surface, but its magnitude varies over surface!

- consider 2^{nd} constraint: **Thermal equilibrium**

  thermal balance: $\frac{dL}{dr} = 4\pi r^2 \varrho \epsilon$

  in terms of radiative flux:

$$\vec{\nabla} \cdot \vec{F} = \varrho \epsilon$$

$$F_{rad} = -\frac{c}{\kappa \varrho} \frac{dP_{rad}}{dr} = -\frac{4}{3} \frac{c}{\kappa \varrho} a T^3 \frac{dT}{dr}$$

  for spherical star.

  for the rotating star, rephrase in terms is $d\phi_{\text{eff}}$

$$\vec{F} = -\frac{4ac}{\kappa \varrho} T^3 \frac{dT}{d\phi_{\text{eff}}} \vec{\nabla} \phi_{\text{eff}}$$

  if the star is thermally stable, then

$$\vec{\nabla} \cdot \vec{F} = 0$$

  but, this is **NOT** full filled:

  take $\vec{F} = K \vec{\nabla} \phi_{\text{eff}}$ then

$$\begin{aligned} \vec{\nabla} \cdot \vec{F} &= \vec{\nabla} K \vec{\nabla} \phi_{\text{eff}} + K \vec{\nabla}^2 \phi_{\text{eff}} \\ &= \frac{dK}{d\phi_{\text{eff}}} \left(\vec{\nabla} \phi_{\text{eff}} \vec{\nabla} \phi_{\text{eff}} \right) + K \vec{\nabla}^2 \phi_{\text{eff}} \end{aligned}$$

  our assumption is T =constant, P =constant, ϱ =const.
  on ϕ_{eff} surface $\rightarrow \frac{dK}{d\phi_{\text{eff}}} = 0$ on surface

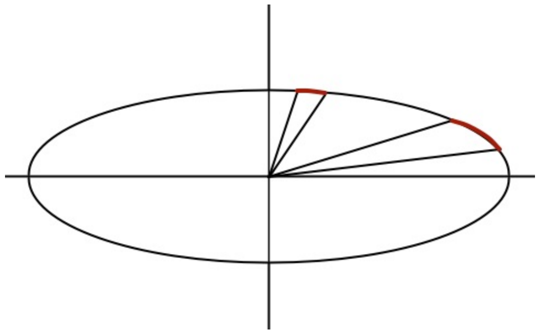
$$\vec{\nabla} \cdot \vec{F} = K \vec{\nabla}^2 \phi_{\text{eff}} = K \left[4\pi G \varrho - 2\Omega \right]$$

  this cannot vanish every where!

  this is the von Zeipel´s paradox.

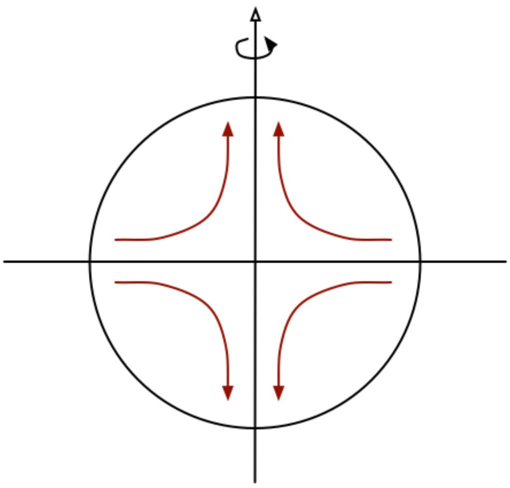
- **Solution:** not in equilibrium $\rightarrow$ **MERIDIONAL MOTIONS**
  - surface area per unit solid angle greater an equator than on pole

- in equilibrium: same flux transported through larger surface area as through smaller one



- temperature drops locally on equator to compensate for that and rises on poles.
- because of bouncy effects warmer gas close to rotational axis begins to **rise** and colder gas at the equator begins to **fall**.

- meridional circulation will **mix** the star



- star is **hotter = brighter** on poles and **colder = darker** on equator

  gravity darkening $\quad T_{\text{eff}} \propto g^{1/4}$

- **Note:** the timescale for meridional circulation is long:

  critical parameters $\quad \beta = \dfrac{E_{rot}}{|E_{\text{grav}}|} = \dfrac{R^3\omega^2}{GM}$

$$E_{\text{rot}} : \quad \Omega^2 R^2 M$$
$$E_{\text{grav}} : \quad \frac{GM^2}{R}$$

$$t_{\rm circ} \approx \frac{t_{KH}}{\beta} = \left(\frac{GM}{R^3\omega^2}\right) t_{KH}$$

- for rapid rotators $\beta = 0.1$

  $$t_{\rm circ} \approx 10^2 t_{KH} \approx 10^9\,\mathrm{yr}$$

  This is larger than main sequence lifetime!

  $\hookrightarrow$ mixing is not efficent

- for slow rotators $\beta \lesssim 0.01$

  $$t_{\rm circ} \approx 10^4 t_{KH} \approx 10^{11}\,\mathrm{yr}$$

  still larger than main sequence lifetime.

- **Note:** this meridional motion may also be induced by tidal perturbations in close binary systems.

11 Magnetic Fields in Stars

A brief deviation into dynamo theory

- relation between current, and electric and magnetic field $\vec{E}$, $\vec{B}$:

$$\vec{j} = \sigma \left(\vec{E} + \frac{\vec{v}}{c} \times \vec{B} \right) \qquad \sigma = \text{electrical conductivity}$$

- chance of B:

$$\frac{\partial \vec{B}}{\partial t} = -c \vec{\nabla} \times \vec{E} \qquad \text{(induction equation)}$$

$$\begin{aligned} \frac{\partial \vec{B}}{\partial t} &= \vec{\nabla} \times \vec{v} \times \vec{B} - \frac{c}{\sigma} \vec{\nabla} \times \vec{j} \\ &= \vec{\nabla} \times \vec{v} \times \vec{B} + \eta \vec{\nabla}^2 \vec{B} \end{aligned}$$

with $\eta = \dfrac{4\pi c^2}{\sigma}$ = diffusion coefficient.

- if we decompose all fields into a mean part and a fluctuating par:

$$\begin{aligned} \vec{j} &= \left\langle \vec{j} \right\rangle + \vec{j}' \\ \vec{E} &= \left\langle \vec{E} \right\rangle + \vec{E}' \\ \vec{B} &= \left\langle \vec{B} \right\rangle + \vec{B}' \end{aligned}$$

then we get: $\left\langle \vec{j} \right\rangle = \sigma \left(\left\langle \vec{E} \right\rangle + \frac{1}{c} \left\langle \vec{v} \right\rangle \times \left\langle \vec{B} \right\rangle + \frac{1}{c} \left\langle \vec{v}' \times \vec{B}' \right\rangle \right)$

$$\frac{\partial \left\langle \vec{B} \right\rangle}{\partial t} = \vec{\nabla} \times \left(\left\langle \vec{v} \right\rangle \times \left\langle \vec{B} \right\rangle \right) + \eta \vec{\nabla}^2 \left\langle \vec{B} \right\rangle + \vec{\nabla} \times \epsilon$$

$$\begin{aligned} \epsilon &= \text{mean electromotive force} \\ &= \left\langle \vec{v}' \times \vec{B}' \right\rangle \end{aligned}$$

- assume ϵ comes from turbulent motions!

  if $\vec{v}'$ and $\vec{V}'$ (the fluctuating parts) are completely uncorrelated then

  $$\left\langle \vec{v}' \times \vec{B}' \right\rangle = \langle \vec{v}' \rangle \times \langle \vec{B}' \rangle = 0$$

- assume $\vec{v} = 0$
  - what remains is:

    $$\frac{\partial \left\langle \vec{B} \right\rangle}{\partial t} = \eta \vec{\nabla}^2 \left\langle \vec{B} \right\rangle$$

    this is a diffusion equation: the magnetic field "diffuses" away: it decays in the absence of dynamo action!
  - time scale: τ

    $$\left\langle \vec{B} \right\rangle \tau^- 1 \approx \eta \left\langle \vec{B} \right\rangle L^{-2}$$

    $$\tau \approx \frac{L^2}{\lambda} \approx \frac{4\pi\sigma L^2}{c^2}$$

Earth:	$L \approx 3 \cdot 10^8$ cm	$\sigma \approx 10^{16}$ esu	$\rightarrow$	$\tau \approx 3 \cdot 10^5$ yr
Sun:	$L \approx 5 \cdot 10^{10}$ cm	$\sigma \approx 10^{17}$ esu	$\rightarrow$	$\tau \approx 10^{11}$ yr
Galaxy:	$L \approx 3 \cdot 10^{20}$ cm	$\sigma \approx 10^{10}$ esu	$\rightarrow$	$\tau \approx 3 \cdot 10^{23}$ yr

- assume $\epsilon = 0$ and $\eta = 0$

  $$\frac{\partial \left\langle \vec{B} \right\rangle}{\partial t} = \vec{\nabla} \times \left(\vec{v} \times \left\langle \vec{B} \right\rangle \right)$$

  this leads to flux freezing!

  the $\vec{B}$-field is compressed just like density

  $$\frac{d \left\langle \vec{B} \right\rangle}{dt} = -\vec{B} \left(\vec{\nabla} \cdot \vec{v} \right) \quad \text{and} \quad \frac{d\varrho}{dt} = -\varrho \vec{\nabla} \vec{v}$$

  with $\dfrac{d}{dt} = \dfrac{\partial}{\partial t} + \vec{v} \cdot \vec{\nabla}$

  $$B = \left| \left\langle \vec{B} \right\rangle \right| = f\varrho \rightarrow \text{scales linearly with } \varrho$$

- Dynamo amplification of field comes from electromotive force ϵ:

  Statistical turbulence theory allows for the ansatz:

  $$\epsilon = \left\langle \vec{v}' \times \vec{B}' \right\rangle = \alpha \left\langle \vec{B} \right\rangle - \eta_T \vec{\nabla}_x \times \left\langle \vec{B} \right\rangle$$

  with η_T being $\frac{1}{3}\langle \vec{v}' \cdot \vec{u}' \rangle \tau$ the turbulent resistivity.

  it turns out $\eta_T \gg \eta$ (molecular resistivity) if α is much larger than nT, α dominates the system and we call it α-Ω dynamo

- **Note:** dynamos can only amplify preexisting fields $\rightarrow$ need a seed field

12 The Early Stages of Stellar Evolution: Proto-stellar Collapse and Pre-Main-Sequence Contraction

- Stars form from gravitationally unstable gas in molecular clouds:
  pre-stellar cloud cores
- Stars (in particular higher-mass stars) typically form as members of a binary or higher order multiple system.
- Stars typically form in clusters (of several hundred objects)

12.1 Proto-stellar Collapse: Jeans mass

- **Gravitational instability:** Jeans criterion
- often the first approach to determine stability properties is to analyze the **linearized** set of equations and derive a **dispersion relation** for the perturbation assumed.
- Linearized equation´s for isothermal self-gravitating fluid:

$$\begin{aligned} \frac{\partial \varrho_1}{\partial t} + \varrho_0 \vec{\nabla} \cdot \vec{v}_1 &= 0 && \text{(continuity)} \\ \frac{\partial \vec{v}_i}{\partial t} &= -\vec{\nabla} c_s^2 \frac{\varrho_1}{\varrho_0} - \vec{\nabla} \cdot \vec{\phi}_1 && \text{(momentum)} \\ \vec{\nabla}^2 \phi_1 &= 4\pi G \varrho_1 && \text{(Poisson)} \end{aligned}$$

- $\vec{\nabla} c_s^2 \varrho_1 \varrho_0^{-1} = \varrho_0^{-1} \vec{\nabla} p_1$ with $p_1 = c_s^2 \varrho_1$ from EOS.
- neglecting viscous effects ($\eta = \xi = 0$)
- equilibrium characterized by ϱ_0 =const. and $\vec{v}_0 = 0$
- Jeans swindle: Poisson´s equation considers only **perturbed potential** (set $\phi_0 = 0$)
- with $\frac{\partial}{\partial t}$[continuity] + $\vec{\nabla}$[momentum] it follows:

  $$\frac{\partial^2 \varrho_1}{\partial t^2} - c_s^2 \vec{\nabla}^2 \varrho_1 - 4\pi G \varrho_0 \varrho_1 = 0$$

  wave equation for $\varrho_1(\vec{x}, t)$

- analyze in Fourier space:

$$\varrho_1(\vec{x},t) = \int d^3k\, A(\vec{k}) e^{i[\vec{k}\vec{x}-w(k)t]}$$

$\frac{\partial}{\partial t} \to iw \qquad \vec{\nabla} \to i\vec{k}$

- dispersion relation:

$$w^2 = c_0^2 k^2 - 4\pi G \varrho_o$$

  - if density ϱ_0 is small $\to$ dispersion relation of sound waves $w^2 = c_s^2 k^2$
  - or small wavelength $\lambda = 2\pi/k$
  - self-gravity acts "strongest" on **large scales** (small k)
    (gravity is long-range force)
  - λ increases/k decreases/ϱ_0 grows: frequency decreases and eventually $w^2 = 0$!
    $\hookrightarrow$ time evolution $\propto e^{\pm\alpha t}$ (if $\alpha^2 = -w^2$) Exponentially unstable.

- Gravitational collapse for wave numbers

$$k^2 < k_J^2 \equiv \frac{4\pi G \varrho_0}{c_s^2}$$

  - Jeans wave-number

$$k_J$$

  - Jeans wave length

$$\lambda_J = \frac{2\pi}{k_J} = \left(\frac{\pi c_s^2}{G\varrho_0}\right)^{1/2}$$

  - Jeans mass

$$M_J = \frac{4\pi}{3}\varrho_0 \left(\frac{\lambda_J}{2}\right)^3 = \frac{\pi}{6}\varrho \left(\frac{\pi c_s^2}{G\varrho_0}\right)^{3/2}$$

    for a spherical perturbation with $\phi = \lambda_J$

$$\begin{aligned} M_J &= \frac{\pi^{5/2}}{6}\left(\frac{R}{G}\right)^{3/2} \varrho^{-1/2} T^{3/2} \\ &= \frac{\pi^{5/2}}{6} G^{-3/2} \varrho^{-1/2} c_s^3 \qquad (c_s^2 = RT) \end{aligned}$$

- Energy of sound wave $E_{\text{sound}} > 0$; gravitational Energy < 0
  Instability sets in, when net energy is negative, i.e. when λ exceeds λ_J.

12.2 Proto-stellar Collapse: Pressure free Collapse and free-fall Time scale

- solve hydrodynamic equations in 1D (keep time dependency!)

$$\begin{aligned} \varrho \frac{d\vec{v}}{dt} &= -\varrho \vec{\nabla}\phi - \vec{\nabla}P && \text{(equation of motion)} \\ 0 &= \frac{d\varrho}{dt} + \varrho \vec{\nabla} \cdot \vec{v} && \text{(continuity)} \\ \vec{\nabla}\phi &= 4\pi G \varrho && \text{(Poisson equation)} \end{aligned}$$

- consider special case: no pressure force!

$$\frac{d\vec{v}}{dt} = -\vec{\nabla}\phi$$

- 1D spherical symmetry:

$$\begin{aligned} \vec{\nabla} &\rightarrow \frac{d}{dr} \\ \vec{\nabla}^2 &\rightarrow \frac{1}{r^2}\frac{d}{dr}\left(r^2 \frac{d}{dr}\right) \end{aligned}$$

- Lagrangian description of spherical mass shells:

$$\begin{aligned} \frac{dM}{dr} &= 4\pi \varrho r^2 \\ \frac{1}{r^2}\frac{d}{dr}\left(r^2 \frac{d\phi}{dr}\right) &= 4\pi G \varrho && |\text{Poisson} \\ \frac{d}{dr}\left(r^2 \frac{d\phi}{dr}\right) &= 4\pi G \varrho r^2 = G\frac{dM}{dr} \\ r^2 \frac{d\phi}{dr} &= GM && |\text{integration} \\ \frac{d\phi}{dr} &= \frac{GM}{r^2} \end{aligned}$$

- in equation of motion:

$$\begin{aligned} \frac{dv}{dt} &= -\frac{d\phi}{dr} = -\frac{GM(r)}{r^2} && |\cdot 2v\,dt \\ 2v\,dv &= -\frac{GM(r)}{r^2} 2v\,dt && |v = \tfrac{dr}{dt} \rightarrow v\,dt = \tfrac{dr}{dt}dt = dr \\ &= -\frac{GM(r)}{r^2} 2\,dr \\ d(v^2) &= -2GM\frac{dr}{r^2} && |\text{integration} \\ v^2 &= 2GM(r_i)\left(\frac{1}{r} - \frac{1}{r_i}\right) + v_i^2 \end{aligned}$$

with integration constant v_i^2; assume cloud initially at rest $v_i = 0$ at radius r_i with Mass $M(r_i)$

$$v(t) = -\left[2GM(r_i)\left(\frac{1}{r(t)} - \frac{1}{r_i}\right)\right]^{1/2}$$

"−" because of collapse (radial vector points away from center)

time dependence disguised in $r(t)$

- use $M(r_i) = \frac{4\pi}{3}\varrho_i r_i^3$ (cloud is originally homogeneous)

$$\frac{dr}{dt} = -\left[\frac{8\pi}{3}G\varrho_i r_i^2\left(\frac{r_i}{r} - 1\right)\right]^{1/2}$$

- substitute $\cos^2\xi = \dfrac{r(t)}{r_i} \rightarrow \dfrac{dr}{dr_i} = 2\cos\xi\sin\xi\, d\xi$

$$\begin{aligned}
r_i 2\cos\xi\sin\xi\, d\xi &= -\left[\frac{8\pi}{3}G\varrho_i r_i^2\left(\frac{1}{\cos^2\xi} - 1\right)\right]^{1/2} dt \\
&= +\left[\frac{8\pi}{3}G\varrho_i\right]^{1/2}\left[\frac{\sin^2\xi}{\cos^2\xi}\right]^{1/2} dt \\
2\cos\xi\sin\xi\, d\xi &= +\left[\frac{8\pi}{3}G\varrho_i\right]^{1/2}\frac{\sin\xi}{\cos\xi}\, dt \\
2\cos^2\xi\, d\xi &= +\left[\frac{8\pi}{3}G\varrho_i\right]^{1/2} dt
\end{aligned}$$

- use $2\cos^2\xi = 1 + \cos 2\xi$:

$$\begin{aligned}
(1 + \cos 2\xi)\; d\xi &= \left[\frac{8\pi}{3}G\varrho_i\right]^{1/2} dt \qquad |\text{integrate} \\
\xi + \tfrac{1}{2}\sin^2 2\xi &= \left[\frac{8\pi}{3}G\varrho_i\right]^{1/2} t + c
\end{aligned}$$

- with initial condition $r = r_i$ (i.e. $\xi = 0$) at $t = 0$ parameter ξ is in general function of **both** r_i and t!
- free-fall time: $r(t) \rightarrow 0$: this means $\xi \rightarrow \pi/2$
  this happens in a finite time: τ_{ff}

$$\frac{\pi}{2} + \underbrace{\frac{1}{2}\sin\pi}_{=0} = \left[\frac{8\pi}{3}G\varrho\right]^{1/2}\tau_{\text{ff}}$$

$$\tau_{\text{ff}} = \sqrt{\frac{3\pi}{32G\varrho_i}} \qquad \textbf{free-fall time}$$

- $\tau_{\rm ff}$ is a function of initial mean density **only**.

  for a homogeneous sphere with initial density ϱ_i
  all fluid elements have the same $\tau_{\rm ff}$
  $\hookrightarrow$ they all arrive at the center at the same time.

- for a homogeneous sphere, the density evolution can be computed analytically:
  - use the continuity equation:

$$\begin{aligned} \frac{d\varrho}{dM} + \varrho \vec{\nabla} \cdot \vec{v} &= 0 \\ \frac{1}{\varrho}\frac{d\varrho}{dt} + \frac{1}{r^2}\frac{d}{dr}\left(r^2 v\right) &= 0 \\ \frac{d}{dt}(\ln \varrho) + \frac{1}{r^2}\frac{d}{dr}\left(r^2 v\right) &= 0 \end{aligned}$$

    recall, the divergence in spherical coordinates is:

$$\vec{\nabla} \cdot \vec{A} = \frac{1}{r^2}\frac{\partial}{\partial r}\left(r^2 A_r\right) + \frac{1}{r\sin\theta}\frac{\partial}{\partial\theta}\left(A_\theta \sin\theta\right) + \frac{1}{r\sin\theta}\frac{\partial}{\partial\Phi}\left(A_\Phi\right)$$

  - with $r/r_i = \cos^2\xi$ it follows:

$$\begin{aligned} r^2 v &= -\frac{\pi}{2\tau_{\rm ff}} r^3 \frac{\sin\xi}{\cos\xi} \\ \text{and} \quad dt &= \frac{4\tau_{\rm ff}}{\pi}\cos^2\xi\, d\xi \end{aligned}$$

    for a general density profile.

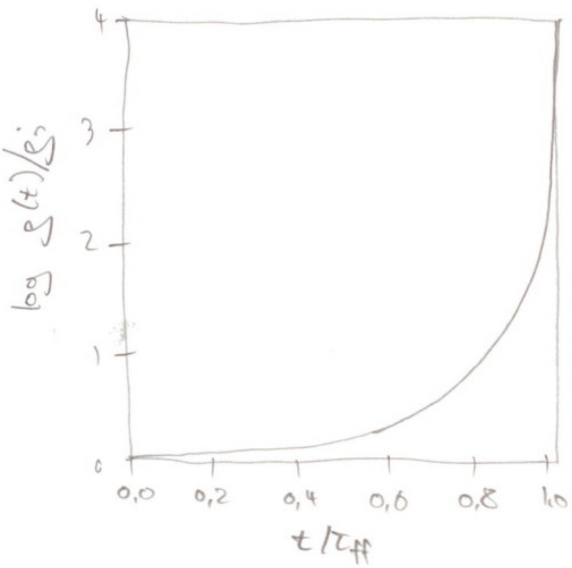
$$d\ln\varrho = 6\tan\xi\, d\xi[1+X]$$

$$\text{with} \quad X = \frac{1}{3}\left[r\frac{d\xi}{dr}\left(3\tan\xi + \cot\xi\right) - \frac{d\ln\tau_{\rm ff}}{d\ln r}\right]$$

  - for $\varrho = \varrho_i$ =constant, **both** $\tau_{\rm ff}$ and ξ are independent of radius $\rightarrow X = 0$ in this case.
  - integration of $d\ln\varrho = 6\tan\xi\, d\xi$ gives

$$\frac{\varrho(t)}{\varrho_i} = \sec^6\xi$$

    if $v_i = 0$



$$\frac{\varrho}{\varrho_i} = 2 \qquad \text{at} \quad t = 0.5\,\tau_{\rm ff}$$
$$\frac{\varrho}{\varrho_i} = 10 \qquad \text{at} \quad t = 0.85\,\tau_{\rm ff}$$

Figure 12.1:

- if $\varrho_i(r)$ decreases outwards, the central parts have smaller $\tau_{\rm ff}$ than outer parts $\rightarrow$ inner parts collapse further $\rightarrow$ density profile steepens $\rightarrow$ non-homologous nature of gravitational collapse.

12.3 gravitational collapse of isolated, isothermal gas spheres

(similarity sln. a la shu, Larson, Penston, Hunter [1977], Whitworth and Ward-Thompson [2001], etc. . . .)

- **hydro equations in 1D and spherical symmetry**

$$\text{closure:} \qquad \varphi = c_s^2 \varrho \tag{12.1}$$

$$\text{equation of motion:} \qquad \frac{dv}{dt} = \frac{\partial v}{\partial t} + v\frac{\partial v}{\partial r} = -\frac{GM(r)}{r^2} - \frac{c_s^2}{\varrho}\frac{\partial \varrho}{\partial r} \tag{12.2}$$

$$\text{continuity:} \qquad \frac{dM}{dt} = \frac{\partial M}{\partial t} + v\frac{\partial M}{\partial r} = 0 \tag{12.3}$$

with $\dfrac{\partial M(r)}{\partial r} = 4\pi r^2 \varrho(r)$

- **SIMILARITY SLN:**
by introducing **one single independent variable** $\xi = c_s t/r$
(following Hunter [1977])

$$\xi = \xi(r,t) \qquad \xi \quad \text{replaces both} \quad r \text{ and } t$$

- set also:

$$\begin{aligned} M(r,t) &= \frac{c_s^3 t}{G} m(\xi) \\ \varrho(r,t) &= \frac{c_s^2}{4\pi G r^2} P(\xi) \\ v(r,t) &= -c_s U(\xi) \end{aligned}$$

- Then (12.1),(12.2),(12.3) reduce to 2 coupled ODE´s for P and U and one algebraic equation defining $m(\xi)$:

$$\begin{aligned} \frac{dU}{d\xi} &= \frac{(\xi U + 1)\left[P(\xi U + 1) - 2\right]}{(\xi U + 1)^2 - \xi^2} \\ \frac{dP}{d\xi} &= \frac{\xi P\left[2 - P(\xi U + 1)\right]}{(\xi U + 1)^2 - \xi^2} \\ m(\xi) &= P\left(U + \frac{1}{\xi}\right) \end{aligned}$$

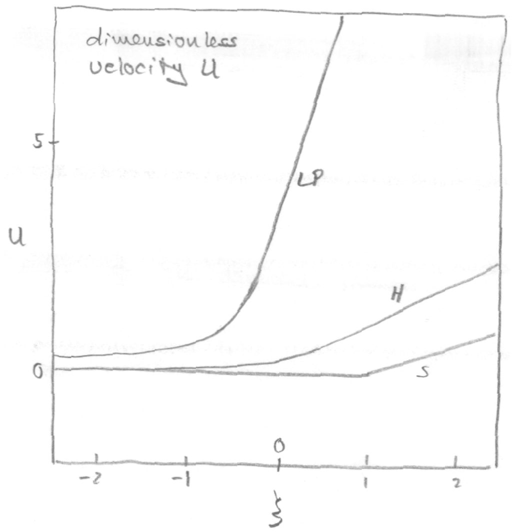


Figure 12.2:

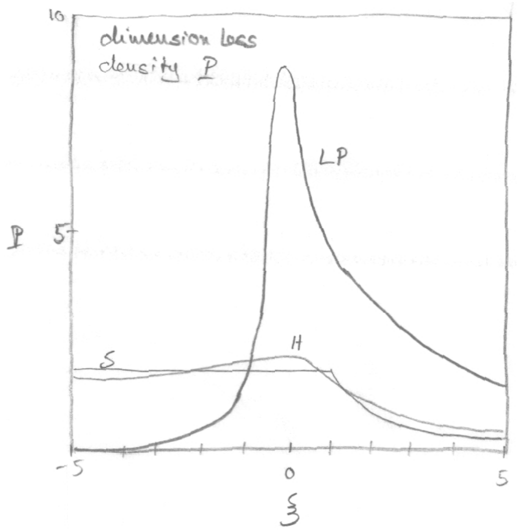


Figure 12.3:

(Figures from Hunter [1977])

- He extended the solutions towards $t \to -\infty$ and $t \to +\infty$
- This sln´s obey $v \to 0$ for $t \to -\infty$
- $t = 0$ is instance of core formation (i.e. occurrence of central singularity)
- originally:
  LP only for $t < 0$
  Sln only for $t > 0$

the portion of each curve with zero slope indicates $\varrho \propto r^{-2}$

- LP = Larson (1969) - Penston (1969) (**) sln:

  $$\text{at } t \to -\infty \quad \varrho = \varrho_0 = \text{const.} \quad v = 0$$

  (collapse of homogeneous sphere from rest)

- S = Shu (1977) (**) sln:

  $$\text{at } t = 0 \quad \text{sis: } \varrho = \frac{1}{r^2} \quad \& \quad v = 0$$

  (collapse of singular isothermal sphere)

- H = Hunter [1977] "in between" solution
- Physical insight can be gained by looking at the behavior in various limits:

  (A) $\xi \to -\infty$

$$\begin{aligned} U &\approx \frac{2}{3}\left(\frac{-1}{\xi}\right) + 45\left[\frac{2}{3} - \exp(Q_0)\right]\left(\frac{-1}{\xi}\right)^3 \\ \ln(\xi^2 P) &\approx Q_0 + \frac{1}{6}\left[\frac{2}{3} - \exp(Q_0)\right]\left(\frac{-1}{\xi}\right)^2 \end{aligned}$$

  with $Q_0 > 0$ constant

  (B) $\xi \approx 0$

$$\begin{aligned} U &\approx U_0 + \xi(P_0 - 2)\xi^2 U_0 + \xi^3\left[(P_0 - 2)\left(1 - \frac{P_0}{6}\right) - \frac{2}{3}U_0^2\right] \\ P &\approx P_0 - \xi^2\left[\frac{1}{2}P_0(P_0 - 2)\right] \end{aligned}$$

  with U_0 and P_0 positive constants

  (C) $\xi \to +\infty$

$$\begin{aligned} U &\approx \sqrt{2m_0\xi} \\ P &\approx \sqrt{M_0/2\xi} \end{aligned}$$

Numbers:	Q_0	U_0	P_0	m_0
LP	0.52	3.28	8.85	46.915
H	11.23	0.295	2.378	2.577
S	$+\infty$	0.0	2.00	0.975

- Conversion into physical quantities (in the various limits)

  $r \ll c_s\,|t|$ and $t < 0$

$$\begin{aligned} v(r,t) &\approx -\frac{2}{3}\frac{r}{-t} \\ \varrho(r,t) &\approx \frac{\exp(Q_0)}{4\pi G}t^2 \end{aligned}$$

  $r \ll c_s\,|t|$ and $t > 0$

$$\begin{aligned} v(r,t) &\approx -\left(\frac{2m_0}{c_s}\right)^{1/2} t^{1/2} r^{-1/2} \\ \varrho(r,t) &\approx \frac{1}{4\pi G}\left(\frac{m_0 c_s^3}{2}\right)^{1/2} t^{-1/2} r^{-3/2} \end{aligned}$$

$r \gg c_s \, |t| \quad \forall t$

$$\begin{aligned} v(r,t) &\approx -c_s U_0 \\ \varrho(r,t) &\approx \frac{c_s^2 P_0}{4\pi G} r^2 \end{aligned}$$

- S and LP sln´s mutually exclude each other

  S: only $0 < \xi < +\infty$:
  claim is that S describes the accretion flow of an isothermal $1/r^2$ envelope onto a recently formed proto-stellar core.

  LP only $-\infty < \xi < 0$:
  transformation of homogeneous sphere into strongly centrally condensed supersonically collapsing cloud.

- $\xi \to -\infty \quad (t \to -\infty)$:
  initial condition of collapse

  $\xi = 0 \quad (t = 0)$
  formation of central zero velocity (and zero radius) core, which for $0 < t \lll -\infty$ grows with $\dot{M} = c_s^3 m_0 / G$

  $\xi \to +\infty \quad (t \to +\infty)$:
  very late stages of collapse when most of the original cloud mass is contained in core.

12.4 Interpretation

- for any stage of collapse prior to core formation, $t < 0$, there will be a region $r \ll c_s(t)$ that is collapsing homologously maintaining uniform density structure and a velocity profile $-v \propto r$.
- there will also be a region $r \gg c_s(t)$ exhibits a $\varrho \propto r^{-2}$ density profile.
- exactly at $t = 0$ the cloud exhibits a $\varrho \propto r^{-2}$ profile everywhere (note, the velocity differs for the different models)
- often core formation, $t > 0$, there is a region at large radii, $r \gg c_s(t)$, that still has $\varrho \propto r^{-2}$ and $v \approx$ constant.
- while near the center, $r \ll c_s(t)$, the cloud structure will change to $-v \propto r^{-1/2}$ and $\varrho \propto r^{-3/2}$

There are important observable differences between S and LP solution, as discussed in previous lectures.

- Shu accretion rate: S-sln maintains

$$M(0,t) = \frac{c_s^3 m_0}{G} t \quad \forall\, t > 0$$

$$M(0,t) = \frac{c_s^3 t}{G} m(\xi)$$

$$\begin{aligned} [c_s] &= \frac{\mathrm{cm}}{\mathrm{s}} \\ [G] &= \frac{\mathrm{cm}^3}{\mathrm{g\,s}^2} \\ [t] &= \mathrm{s} \end{aligned}$$

- "inside-out" collapse: rarefaction wave moves outwards with speed of sound c_s
- at $t = 0$ LP & S differ in the velocity field:

$$\begin{aligned} s &\rightarrow -v = 0 \\ LP &\rightarrow -v \gg c_s \end{aligned}$$

Note: compare Shu rate with Bondi-Hoyle-Lyttleton accretion rate

- Bondi accretion rate:

$$\dot{M}_B = \lambda_c\, 4\pi\varrho_\infty \frac{(GM)^3}{c_\infty^3}$$

$$\begin{aligned} \lambda_c &= \tfrac{1}{4}e^{3/2} = 1.12 \\ \varrho_\infty &= \varrho \text{ at } \infty \\ c_\infty &= c_s \text{ at } \infty \end{aligned}$$

- Shu rate:

$$\dot{M}_s = \frac{c_s^3}{G} m_0$$

$$m_0 = 0.95a$$

- if we set $c_s = c_\infty$, and if we set $M = \dot{M}t$, and if we identify $4\pi G\varrho_\infty = t^{-2}$ with the inverse square of the free-fall time, then we see.

$$\begin{aligned} \dot{M}_B &= \lambda_c \frac{1}{t^2 G}\frac{G^2}{c_\infty^3}\dot{M}_B^2 t^2 \\ 1 &= \lambda_c \frac{G}{c_\infty^3}\dot{M}_B \\ \dot{M}_B &= \frac{1}{\lambda_c}\frac{c_s^3}{G} \approx \dot{M}_s \quad \text{as} \quad \lambda_c^{-1} m_0 \end{aligned}$$

- Bondi problem:
  spherical accretion of gas by a gravitating point mass M.
  (self-gravity of gas is ignored)

- Hoyle-lyttleton:
  if star moves with v_x through the medium

$$\dot{M} = \lambda_* 4\pi\varrho_\infty \frac{(GM)^2}{(v_*^2 + c_\infty^2)^{3/2}}$$

- **in REALITY:** NO SIS $\Rightarrow$ time varying $\dot{M}$

example 1:

multiple power law:

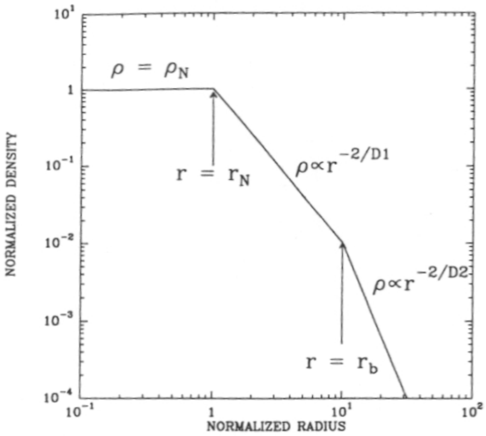


Figure 12.4: Density vs. radius

compare, e.g. with density profile of B68 or the sample of pre-stellar coves by Bacmann et al. (2000)(**)

resulting accretion rates

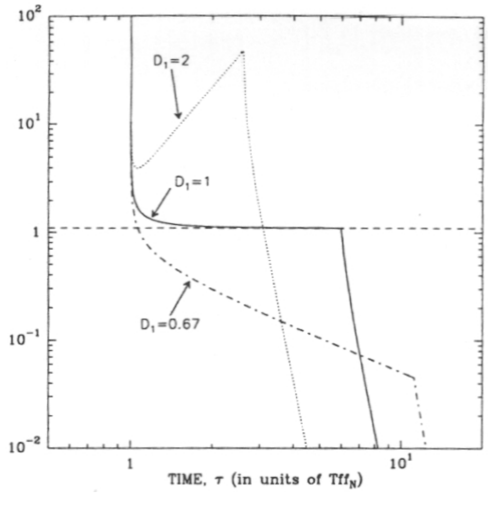


Figure 12.5: Accretion rate

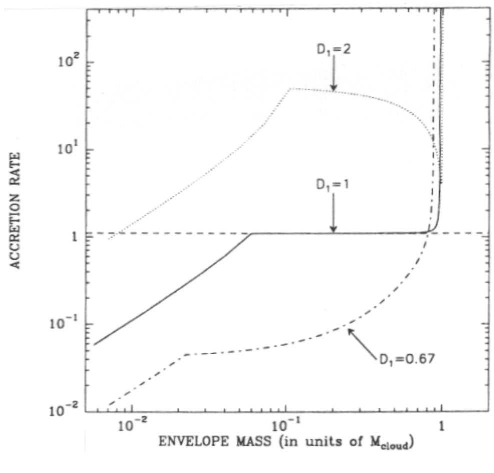


Figure 12.6: Absolute value of the accretion rate

(from Hendriksen et al. 1997 (**))

example 2: Plummer sphere (modeled to fit LI544)

generalized Plummer density distribution

$$\varrho(r, t = 0) = \varrho_{core} \left[\frac{r_{core}}{(r_{core}^2 + r^2)^{1/2}} \right]^\eta$$

original Plummer sphere: $\eta = 5$

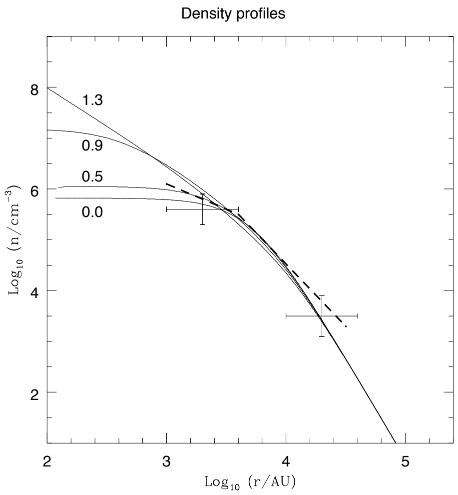


Figure 12.7: Volume-density profiles

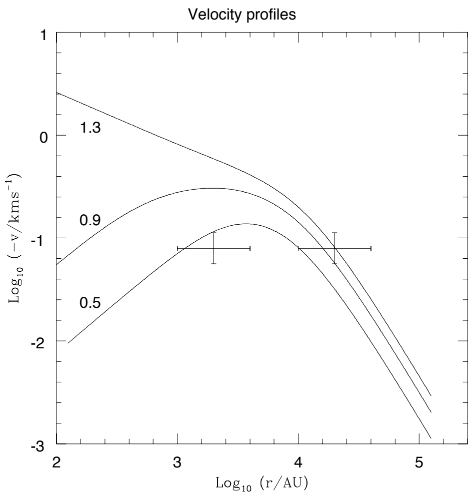


Figure 12.8: Inward radial velocity profiles

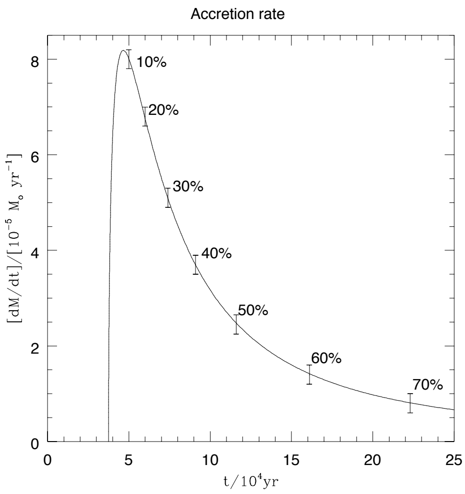


Figure 12.9: Accretion rate

(from Whitworth and Ward-Thompson [2001])

12.5 Models of 1D collapse

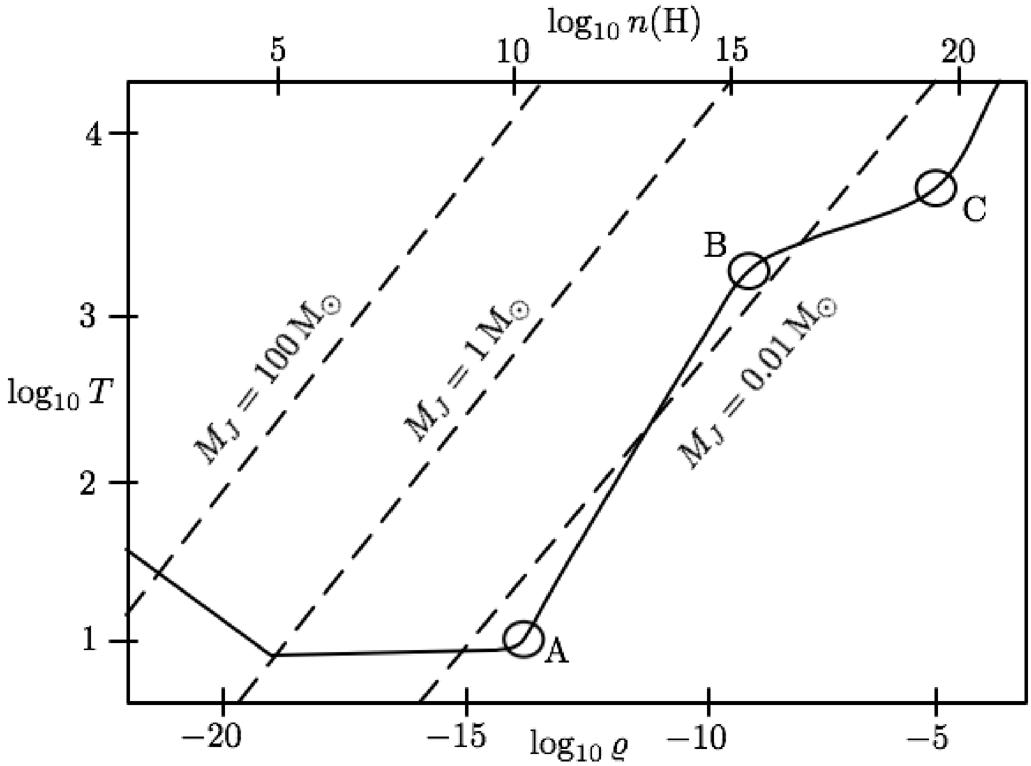


Figure 12.10: Effects of EOS: from Bodenheimer 1980 (**)

$\varrho - T$ condition in the center of a $3\,\mathrm{M}_\odot$ proto-star

(A) end of isothermal phase

(B) onset of H_2 dissociation

(C) end of H_2 dissociation, begin of final adiabatic core

12.6 Collapse in 3D$\longrightarrow$ binary formation

- almost **ALL** stars form as part of a binary or higher-order multiple system.
- pre-stellar coves have angular momentum ($J \approx 10^{20} - 10^{22}\,\mathrm{cm}^2/\mathrm{s}$)
- formation of rotationally supported disk during collapse
- further proto-stellar mass accretion via viscous transport processes (see next lectures)
- often disk "too massive"
  $\longrightarrow$ grav. instability
  $\longrightarrow$ binary formation by disk fragmentation
- for a recent review see Bodenheimer et al (2000, PP IV)(**)

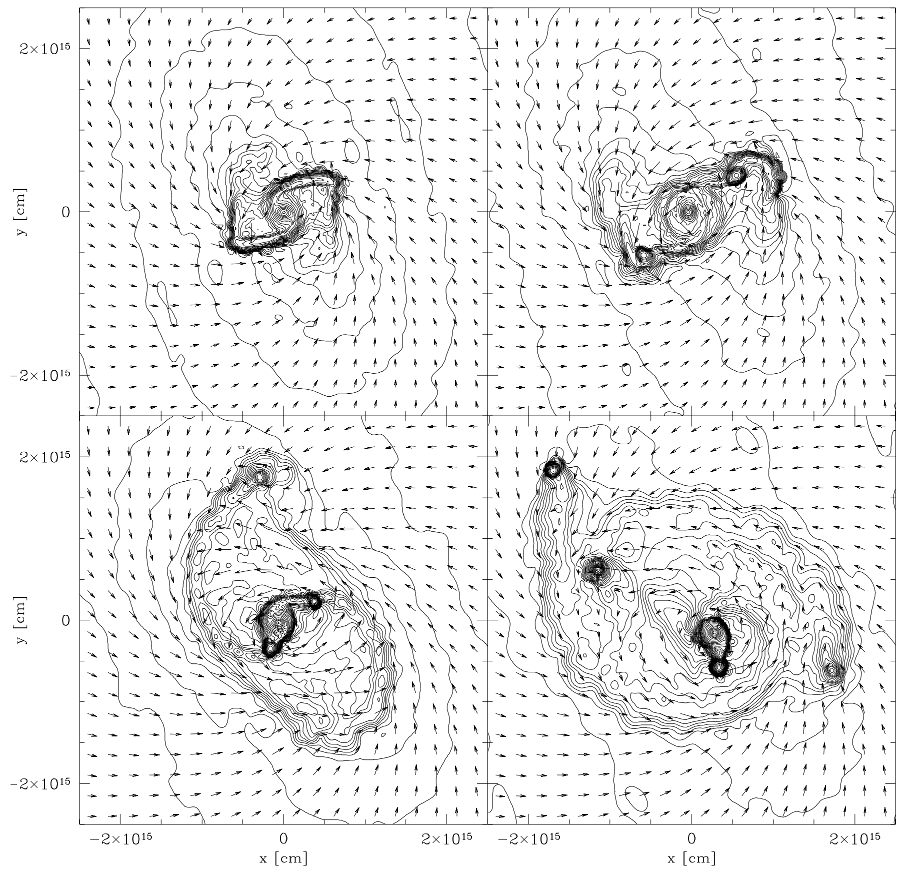


Figure 12.11: Evolution of the high-beta case with the SPH code. [see Burkert et al., 1997, Fig. 4]

13 Evolution off the Main Sequence

13.1 Role of electron degeneracy

- onset of degeneracy provides pressure support that allows stability at (almost) any temperature.
- when degeneracy occurs, temperature "decouples" from hydrostatic balance.
- first we need to consider **electron** degeneracy, only for very compact objects (when e^- are "pushed into" p^+, to give n^0) we need to also take neutrons degeneracy into account (neutron stars)
- recall: Fermi energy

  $$E_F \approx \frac{p_F^2}{2m_E}$$

  to estimate p_F:

  $$n_e = \frac{\varrho}{m_p} \stackrel{!}{=} \frac{8}{3}\frac{\pi}{h^3}p_F^3 \quad \text{(electron density)}$$

  $$p_F^3 = \frac{3}{8}\frac{h^3\varrho}{\pi m_p}$$

  fully ionized

  $$n_e = n_p = n = \frac{\varrho}{m_p}$$

- critical temperature when

  $$\begin{aligned} E_F &\approx E_{\text{thermal}} \\ \frac{p_F^2}{2m_e} &\approx kT \end{aligned}$$

  $$kT \approx \left(\frac{3h^3}{8\pi m_p}\right)^{2/3}\frac{1}{2m_e}\varrho^{2/3}$$

  critical temp:

  $$T_c \approx 3\cdot 10^5\, K\left(\frac{\varrho}{1g/cm^3}\right)^{2/3}$$

recall for hydrostatic balance:

$$T_c \approx 2 \cdot 10^6\, K \left(\frac{M}{1\, M_\odot}\right)^{2/3} \left(\frac{\varrho}{1\, g/cm^3}\right)^{1/3}$$

- so star contracts until either **degeneracy** is reached, or **nuclear burning** occurs:

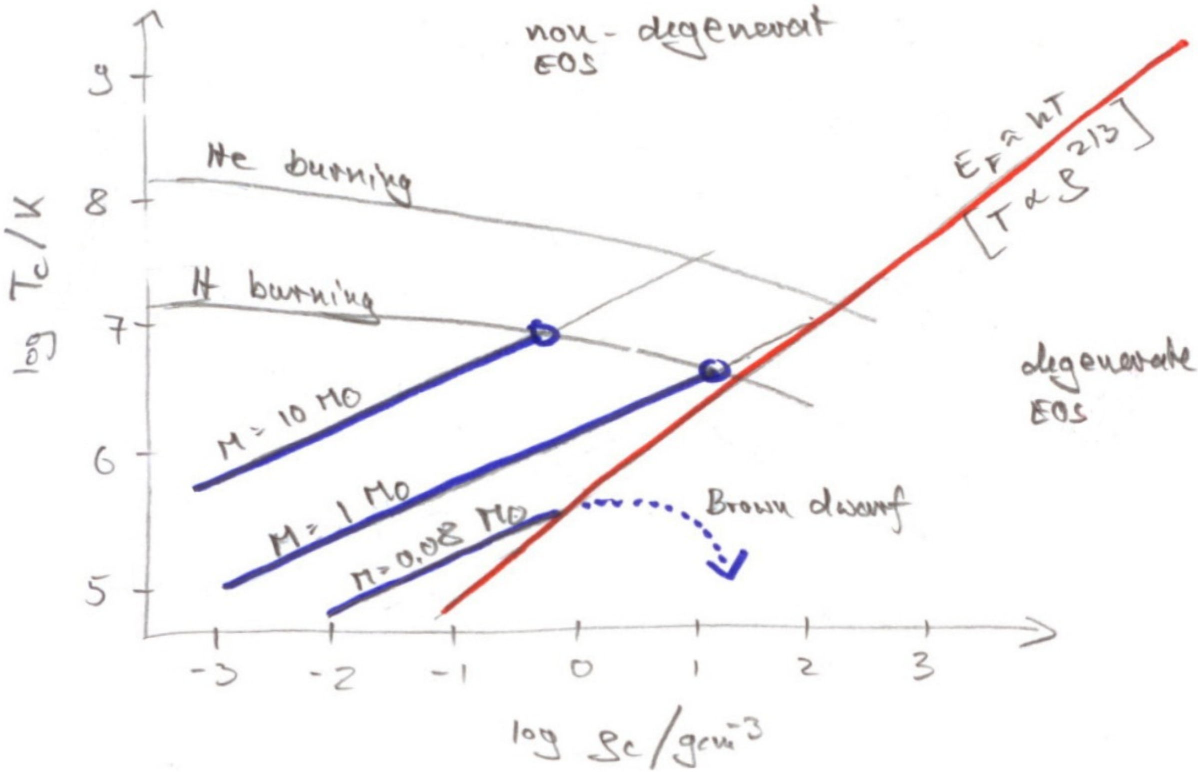


Figure 13.1: non-degenerate/degenerate EOS. [for more details see Iben, 1985]

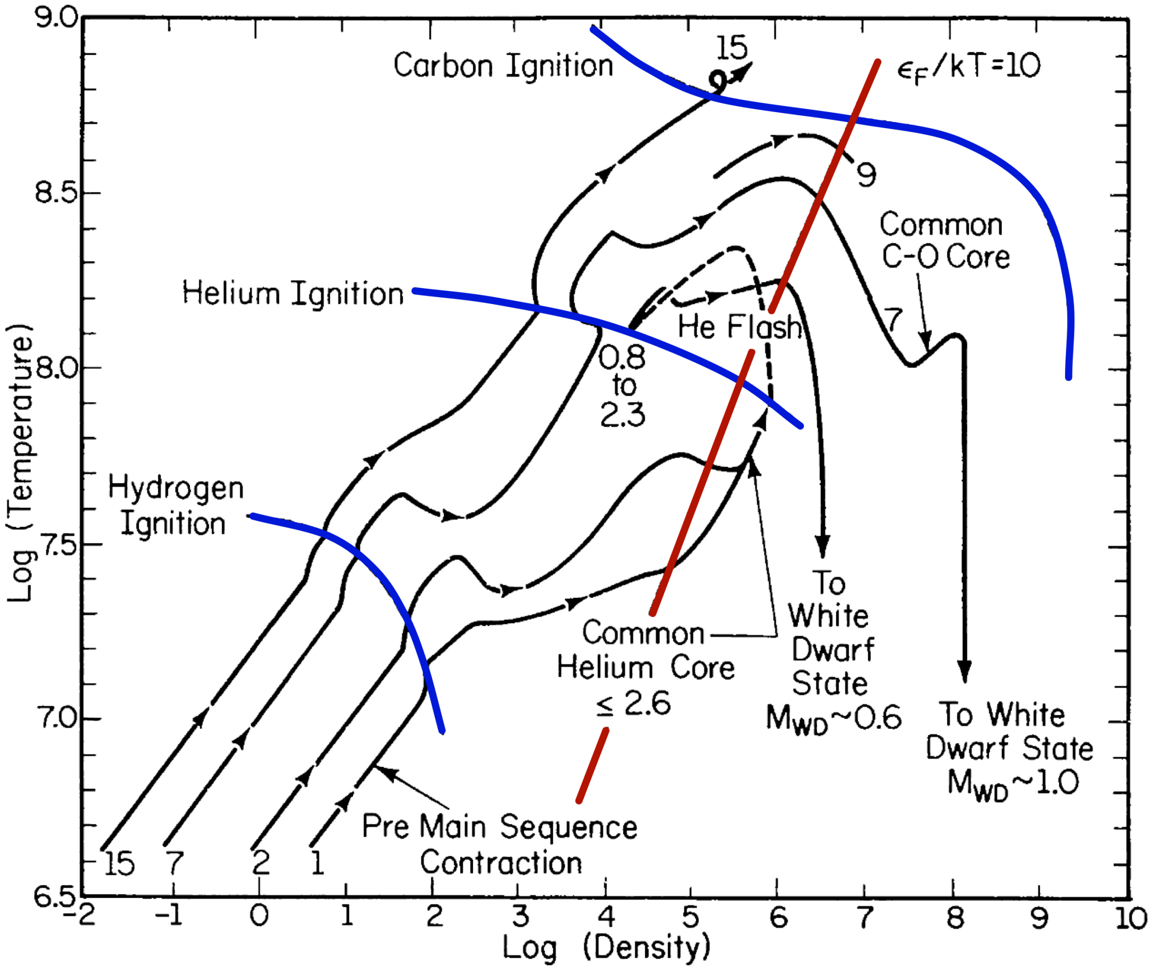


Figure 13.2: [Iben, 1985, Figure 2]

- Stars with initial mass below $\approx 10\,M_\odot$ will turn into white dwarfs after heavy mass loss.
- low-mass stars develop e^--degeneracy core before He burning starts (He burning then may occur in degenerate phase)
- low-to intermediate-mass stars can burn He and develop e^--degenerate core made of carbon and oxygen (C,O core)

- recall evolution **ON** main sequence:

  for $M \gtrsim 1\,\mathrm{M}_\odot$: CNO cycle produces energy

  $$L \propto M^3 \mu^4$$

  as H burns into He, the mean molecular weight μ increases, as $M \approx$ constant

  $$\mu \uparrow \quad \Rightarrow \quad L \uparrow$$

  because $L_{\mathrm{CNO}} \propto T^{20}$ the central temperature increase only slightly to match the growing demand in luminosity

  $$T \approx \text{constant}$$

- **evolution OFF main sequence:**

  as H is depleted, star contracts and heats up, further evolution again mass dependent

  - low masses: e^--degeneracy kicks in before onset of He burning
  - intermediate masses: He burning
  - higher masses: C/O burning
  - up to Fe and SN

- post main-sequence evolution is very diverse, depending on initial stellar mass:

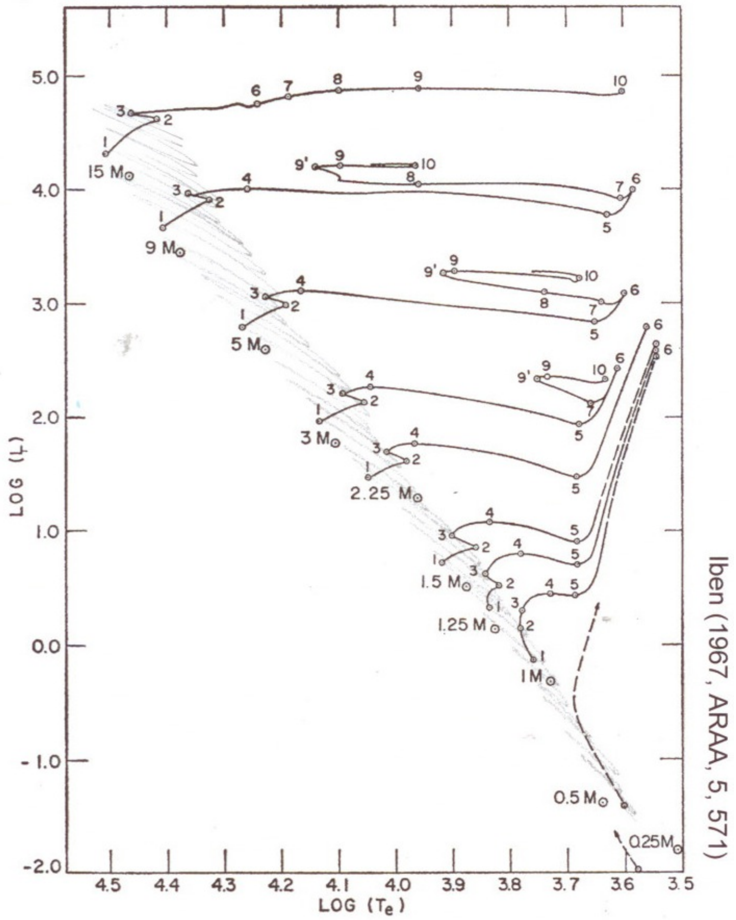


FIG. 3. Paths in the H-R diagram for metal-rich stars of mass $(M/M_\odot)=15$, 9, 5, 3, 2.25, 1.5, 1.25, 1, 0.5, 0.25. Units of luminosity and surface temperature are the same as in Figure 1. Traversal times between labeled points are given in Tables III and IV. Dashed portions of evolutionary paths are estimates.

TABLE III

STELLAR LIFETIMES (yr)[a]

Mass ($M_\odot$) \ Interval ($i-j$)	(1–2)	(2–3)	(3–4)	(4–5)	(5–6)
15	1.010 (7)	2.270 (5)		7.55 (4)	
9	2.144 (7)	6.053 (5)	9.113 (4)	1.477 (5)	6.552 (4)
5	6.547 (7)	2.173 (6)	1.372 (6)	7.532 (5)	4.857 (5)
3	2.212 (8)	1.042 (7)	1.033 (7)	4.505 (6)	4.238 (6)
2.25	4.802 (8)	1.647 (7)	3.696 (7)	1.310 (7)	3.829 (7)
1.5	1.553 (9)	8.10 (7)	3.490 (8)	1.049 (8)	≥2 (8)
1.25	2.803 (9)	1.824 (8)	1.045 (9)	1.463 (8)	≥4 (8)
1.0	7 (9)	2 (9)	1.20 (9)	1.57 (8)	≥1 (9)

[a] Numbers in parentheses beside each entry give the power of ten to which that entry is to be raised.

Figure 13.3: Paths in H-R diagram for metal-rich stars. [Iben, 1967, Figure 3]

- phase 1-2:

  corresponds to He burning phase = main sequence
- phase 2-3:

  rapid phase: contraction of stellar core as H is rapidly depleted
- phase 3-6:

  correspond to **RED GIANT BRANCH**:
  * continuous expansion of envelop
  * reddening of surface to cooler temperatures

13.2 Lower-MS Stars

example: $2.25\,\mathrm{M}_\odot$ star Iben [1985]

- core completely convective $\rightarrow$ is always well mixed
  $\rightarrow$ H burning stops when
  $\approx 0.2\,\mathrm{M}_\odot$ He has been produced.
- now: He **core contracts:**

  $\hookrightarrow$ T_c increases and H ignites in **SHELL** around core
  $\hookrightarrow$ Star moves up red giant branch.
- He flash ignites (off center)
  - this is an almost explosive event (He ignition under electron degenerate conditions)
  - released energy goes into heat T increases, **but** because of e^--degeneracy the pressure is not increased!
    $\hookrightarrow$ star **cannot** cool by expansion!
  - as T increases rapidly, the energy production in 3α process increases dramatically (eventually the outer convective zone depends and begins to transport away the energy released) [1]
  - He core burns quietly. star expands dramatically (and reddens) [2]
  - Red giant phase
- while He burns (quietly) in core C and O build up and eventually He burning ceases.
  - He shell burning still continues

[1] also He is transported outwards
$\hookrightarrow$ first dredge up
[2] also H burning shell expands $\rightarrow$ gets cooler
$\hookrightarrow$ L decreases!

– **Asymptotic Giant BRANCH** (He shell dominates)

- there is the possibility for He shell flashes (and additional dredge ups) and for higher mass objects: thermal pulses that can revitalize H burning outer shells.
- now heavy mass loss in post-AGB phase (line driven winds)
  $\hookrightarrow$ formation of planetary nebulae
  $\hookrightarrow$ remnant $\approx 0.6\,\mathrm{M}_\odot$ CO white dwarf

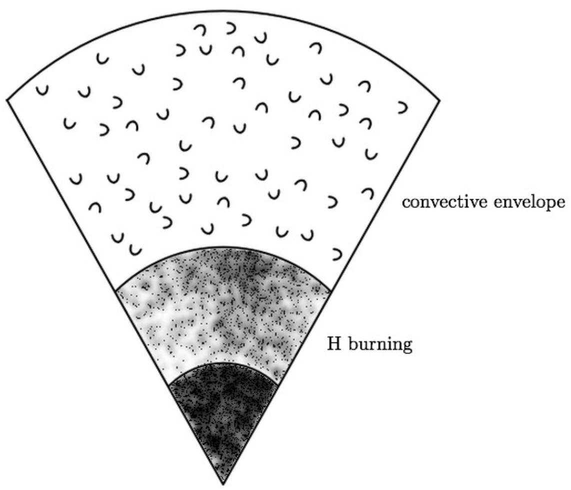


Figure 13.4:

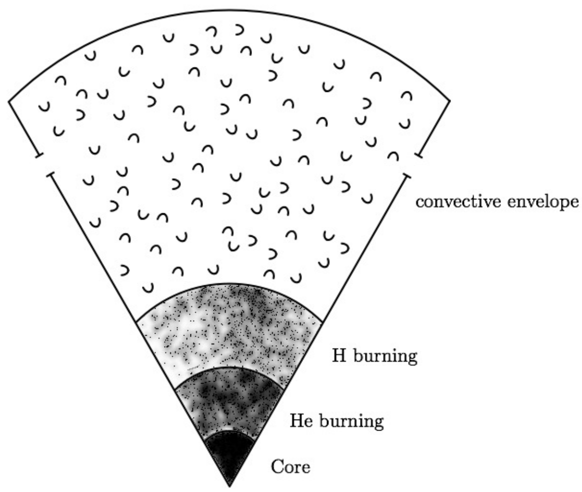


Figure 13.5:

- very similar: **Evolution of** $1\,\mathbf{M}_\odot$ star

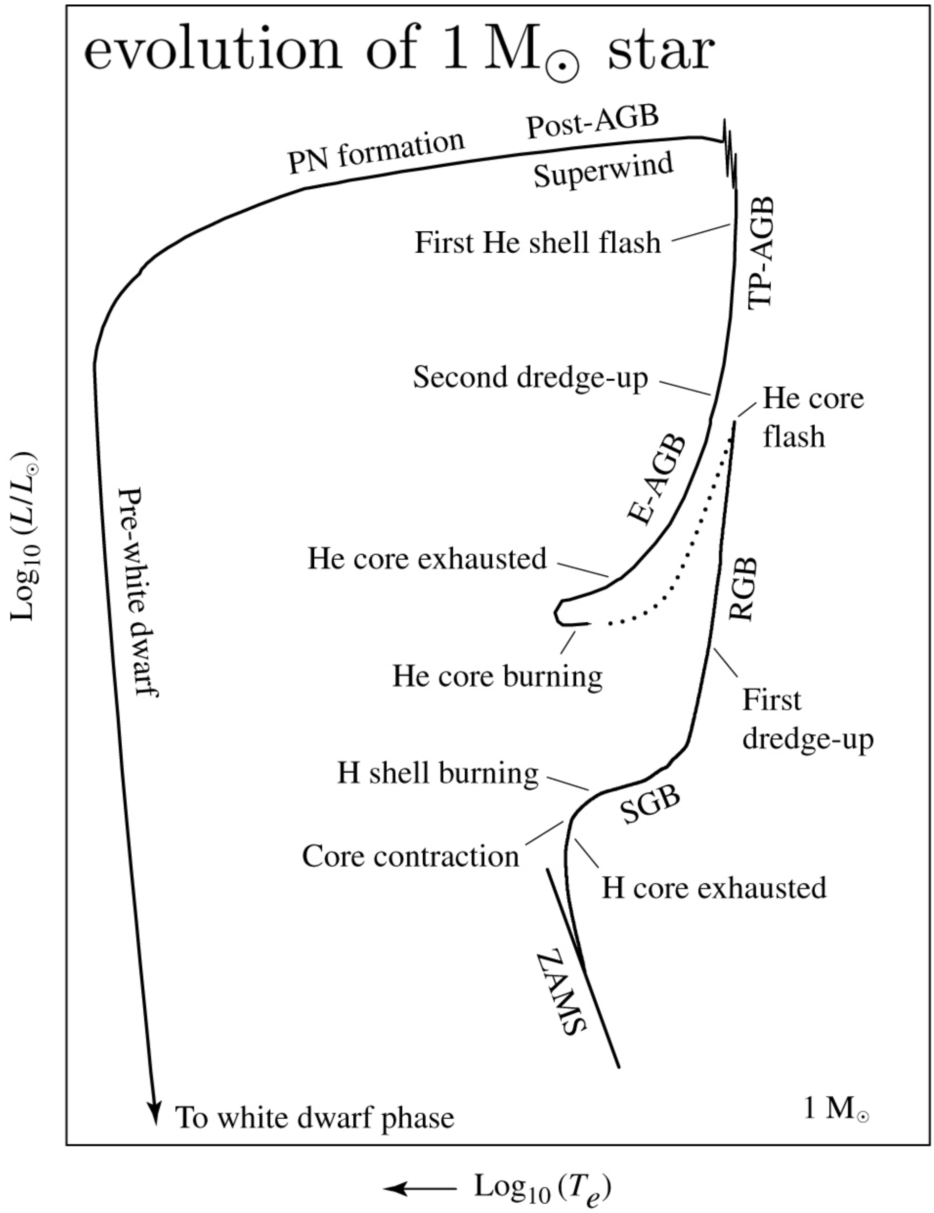
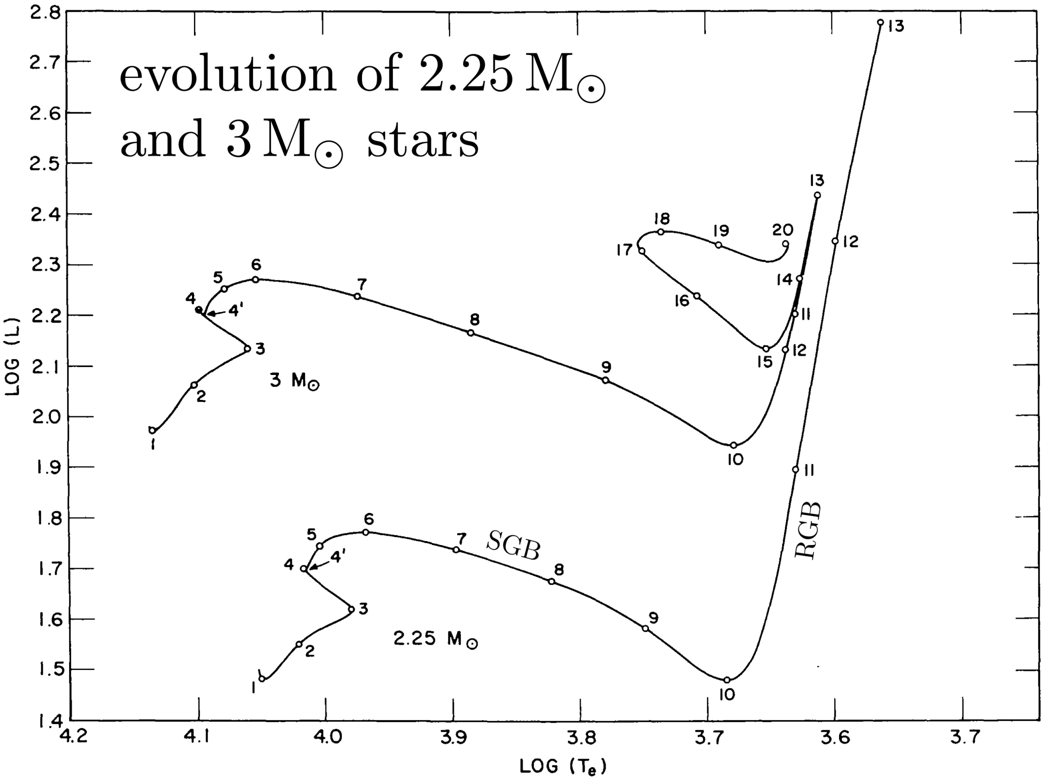


Figure 13.6: Evolution of $1\,\mathrm{M}_\odot$ star. [Carroll and Ostlie, 2006, Chapter 13]



RGB = red giant phase, SGB = snb giant phase

Figure 13.7: Evolutionary tracks in the Hertzsprung-Russell diagram for model stars of mass $2.25\,\mathrm{M}_\odot$ and $3\,\mathrm{M}_\odot$. [see Iben, 1985, Figure 1]

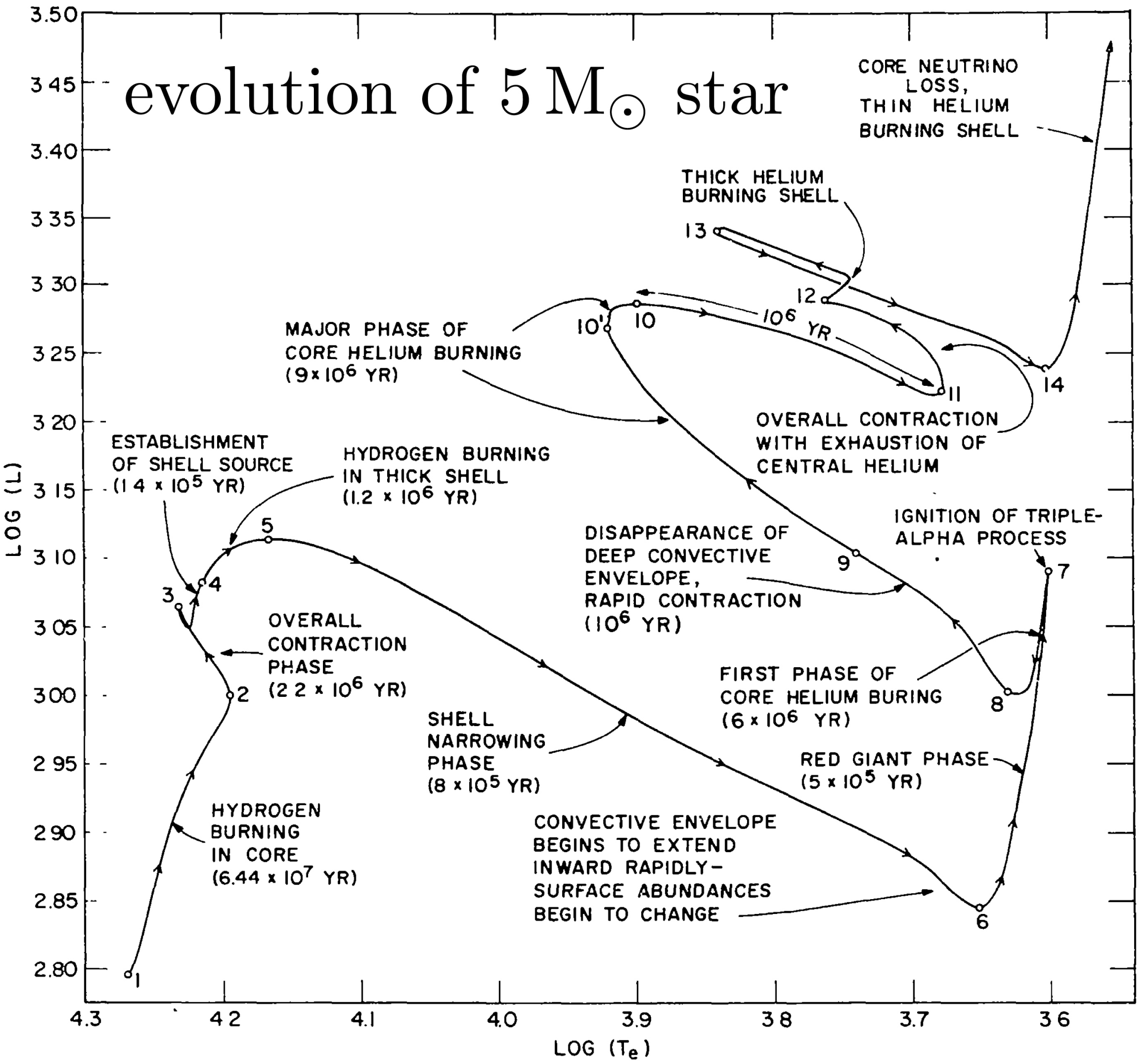


Figure 13.8: Evolution of 5 $M_\odot$ star.
[after Iben, 1985, Figure 3], [see also Carroll and Ostlie, 2006, Figure 13.5]

- similar to the 1 $M_\odot$ star, but now a well-established **HORIZONTAL BRANCH** phase occurs.

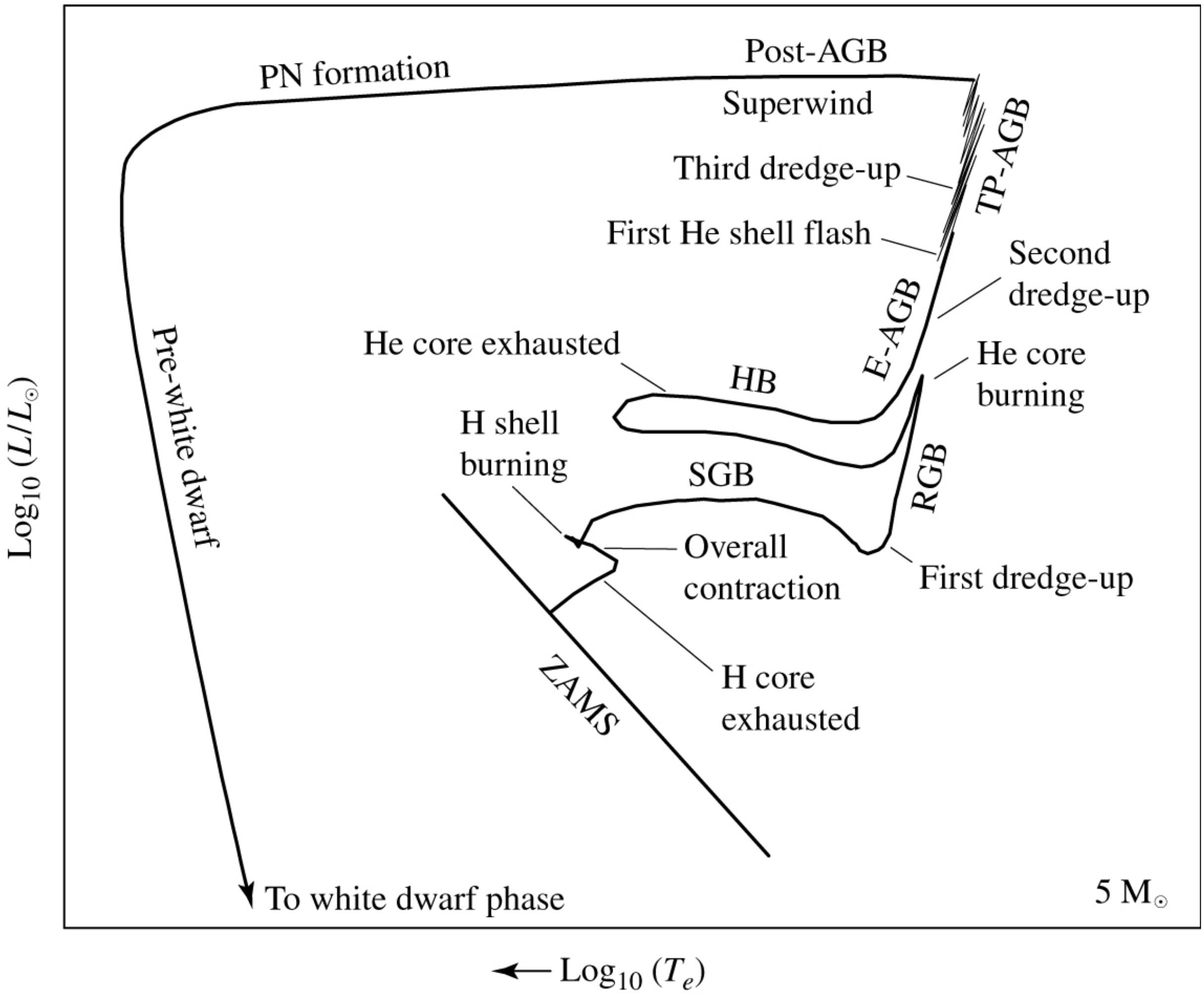


Figure 13.9: [Carroll and Ostlie, 2006, Chapter 13]

The HB occurs when He **burns** in the core which becomes **convective** (no longer e^--degenerate).
HB is the analog to the fully convective CNO H-burning cores of upper-main sequence stars.

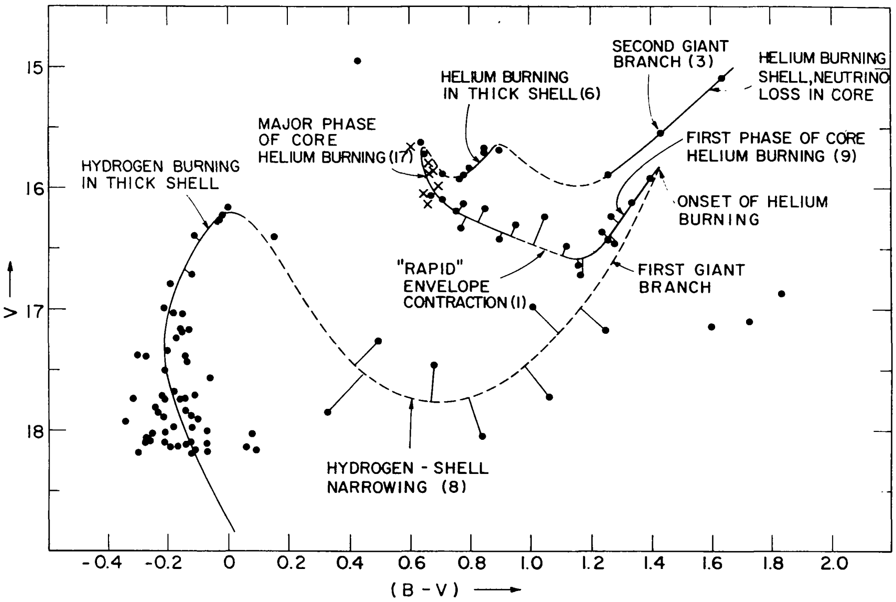


Figure 13.10: Eye fit to data. The fit is similar to the $5\,M_\odot$. [see Iben, 1985, Figure 4]

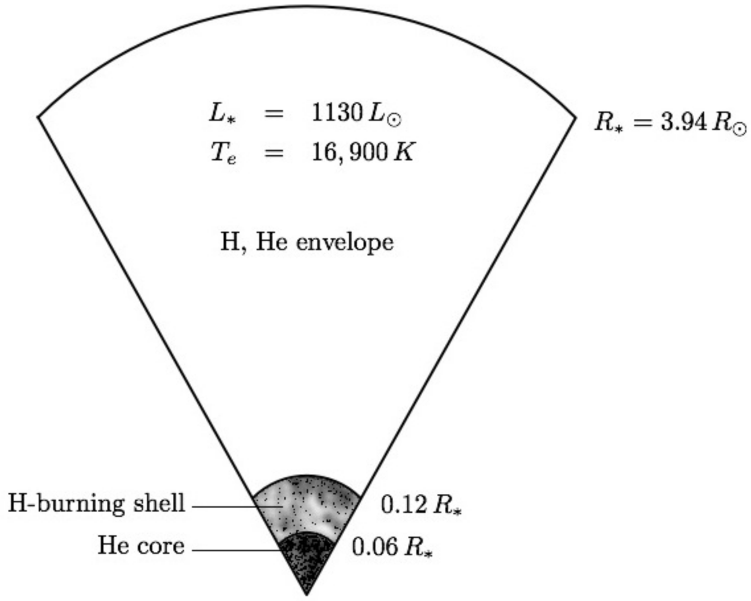
- when the evolution of the star reaches the blue-ward point of the HD, then μ has changed (dominated by C and O) so that the core needs to contract again to produce required luminosity
  $\hookrightarrow$ envelope expands and cools
  $\hookrightarrow$ star moves red-wards.

- in red part of HB core helium is exhausted.
  $\hookrightarrow$ core contracts (just like in sub giant phase: SGB)
  $\hookrightarrow$ as stars more through HB, instabilities and pulsation may occur.

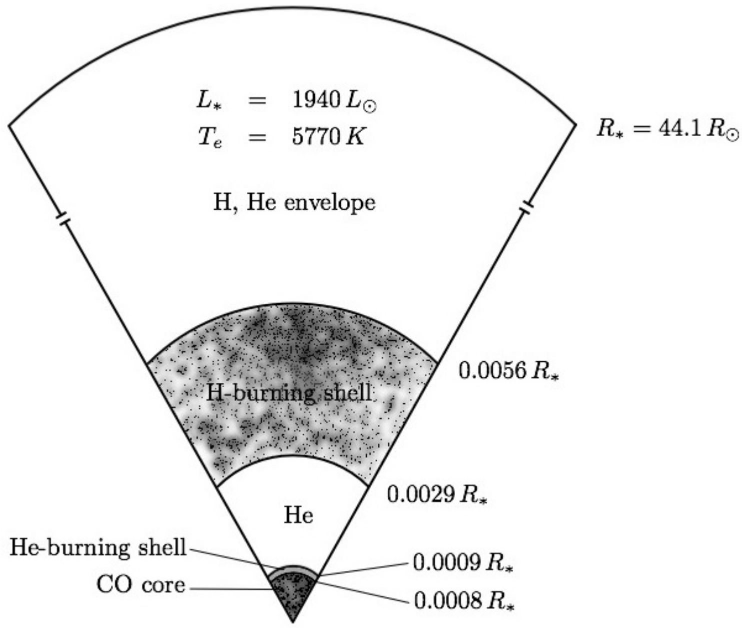
- then the (early) AGB comes, where the luminosity is generated by He shell burning (similar to H shell burning at RGB)

- intermediate-mass stars then may go through thermal pulses: narrow He shell may turn and off:
  - as star contracts, dormant H shell wakes up.
  - H shell produces He ashes that rain down on He shell.
  - as mass of He shell increases it may become slightly degenerate.
  - then, when the temperature at the base of the shell increases sufficiently, He may ignite again and a He flash occurs
    (analogous to the earlier He flash of low-mass stars).
  - eventually He burning ceases and H burning recovers and the process repeats.

- structure of star during RGB, shortly after H-burning shell ignited.



- structure of star during He flash



[Carroll and Ostlie, 2006, Chapter 13]

- thermal pulses at end of AGB phase:

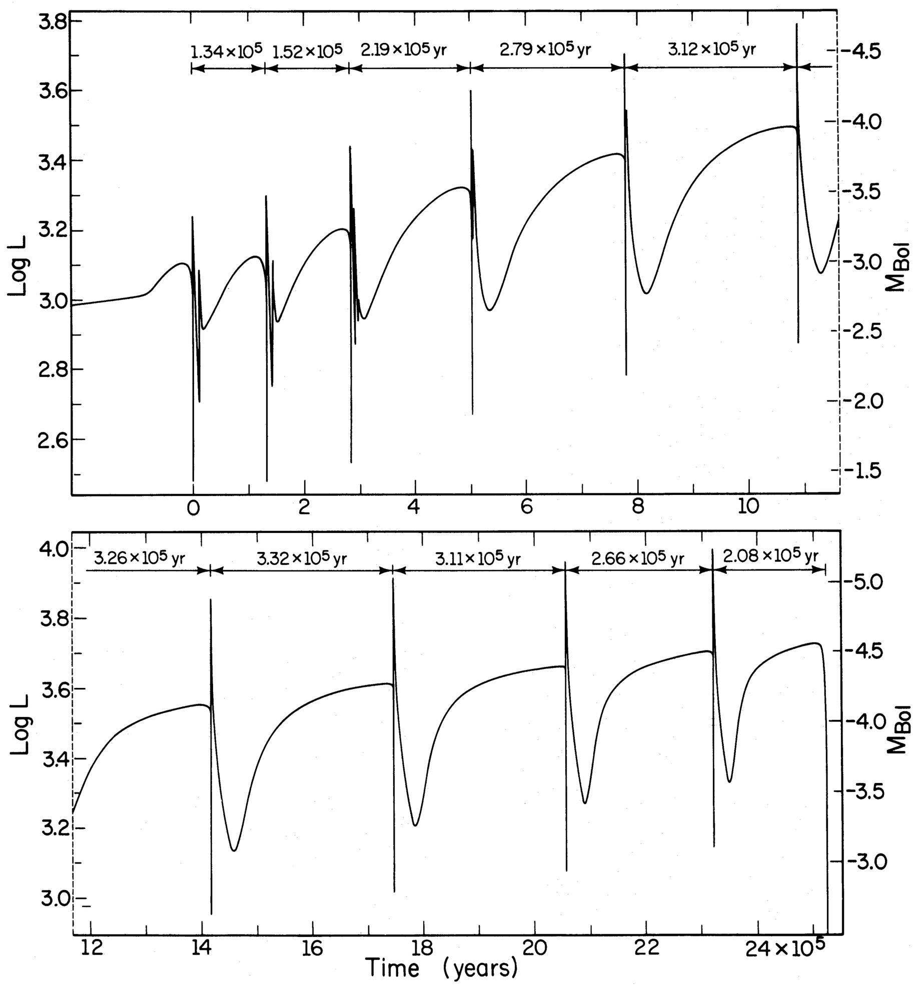


Figure 13.11: Time dependence of luminosity and bolometric magnitude during the thermally pulsing phase for a model of mass $0.6\,M_\odot$. [see Iben, 1982, Figure 2]

- during He flashes, a convection zone occurs between He and H burning shell; this zone may merge with the convection zone of the envelope
  $\hookrightarrow$ carbon may be mixed up all the way to the stellar surface

- this is the “third dredge up” which leads to carbon-rich giants (C spectral type)

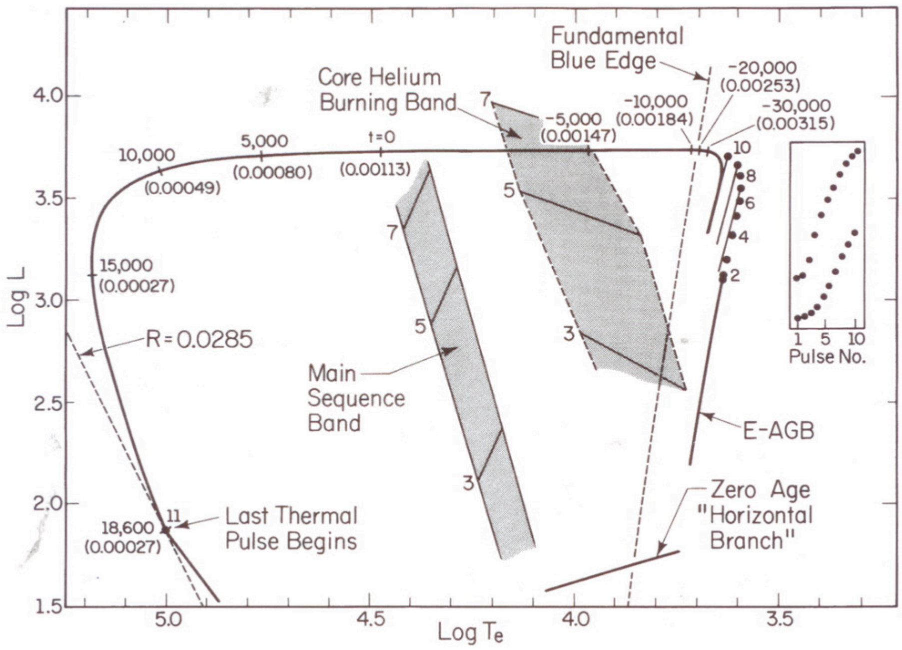


Figure 13.12: Evolutionary track in the H-R diagram of an 0.6 $M_\odot$ model with a carbon-oxygen core. [after Iben, 1985, Fig. 16], [see also Carroll and Ostlie, 2006]

- AGB stars are know to lose mass at high rate (up to $\dot{M} = 10^{-4}\,M_\odot/\mathrm{yr}$)

  $T_{\mathrm{surface}} \approx 300\,\mathrm{K}$ they are quit cool

  $\hookrightarrow$ dust can form

  - silicates in O-rich environment
  - graphite in C-rich stars

- this dust may couple to radiation and this radiation may drive the large mass loss. (not well understand, maybe thermal pulses)

- as the envelope expands, it eventually becomes optical thin.
  thus exposing the central star, which typically shows spectrum of F-type or G-type super-giant.

  the evolutionary track moves more or less horizontally to the blue

- eventual expelled envelope + central white dwarf make up a **planetary nebula** (PN)

13.3 Post main sequence evolution of very-massive stars

- stars initially more massive than $\sim 8 \ldots 10\,\mathrm{M}_\odot$ end their lives in supernova explosions.
- before that, they go through several higher-element nuclear burning phases and several mass-loss phases.
  - **luminous blue variables** (LBVs)
    - have surface temperatures, luminosities and masses
      $$\begin{aligned} T &\approx 15,000\,\mathrm{K} - 30,000\,\mathrm{K} \\ L &\gtrsim 10^6\,\mathrm{L}\odot \\ M &\geqslant 85\,\mathrm{M}_\odot \end{aligned}$$
      $\hookrightarrow$ sit in upper left of H-R diagram.
    - these stars have huge mass loss rates and they are rapid rotators.
  - **Wolf-Rayet stars** (WR)
    - are closely related to LBVs
    - there are $\sim 1000 - 2000$ WRs in the Galaxy
    - have $T_{\mathrm{eff}} \approx 25.000\,\mathrm{K} - 100.00\,\mathrm{K}$ and show very strong emission lines
    - have huge mass loss rate $10^{-5}\,\mathrm{M}_\odot/\mathrm{yr}$ with wind speeds $\gtrsim 800\mathrm{km/s}$
    - have masses $\gtrsim 20\,\mathrm{M}_\odot$
    - three classes:

WC	have emission lines of C and He (**no** N & H)
WN	have N and He
WO	have strong O-lines

  - there are also **blue super-giants** (BSG), **red super-giants** (RSG), and **OF stars** (O super-giants with pronounced emission lines)
- the general evolutionary sequence is as follows:

$$\begin{array}{lllll} 10 < M/\mathrm{M}_\odot < 20: & O \rightarrow RSG \rightarrow BSG \rightarrow \mathbf{SN} \\ 20 < M/\mathrm{M}_\odot < 25: & O \rightarrow RSG \rightarrow WN \rightarrow \mathbf{SN} \\ 25 < M/\mathrm{M}_\odot < 40: & O \rightarrow RSG \rightarrow WN \rightarrow WC \rightarrow \mathbf{SN} \\ 40 < M/\mathrm{M}_\odot < 85: & O \rightarrow Of \rightarrow WN \rightarrow WC \rightarrow \mathbf{SN} \\ 85 < M/\mathrm{M}_\odot : & O \rightarrow Of \rightarrow LBV \rightarrow WN \rightarrow WC \rightarrow \mathbf{SN} \end{array}$$

- **supernovae**
  - supernovae describe the explosive, sudden mass loss (very-)massive stars experience at the end of their lives.
  - there are two classes of SN

Type I:	no hydrogen	lines
Type II:	strong H	lines

    Type I´s are sub-classified according to other spectral features:

Type Ia:	strong Si II	line (615 nm)
Type Ib:	strong He I	line
Type Ic:	no He I	line

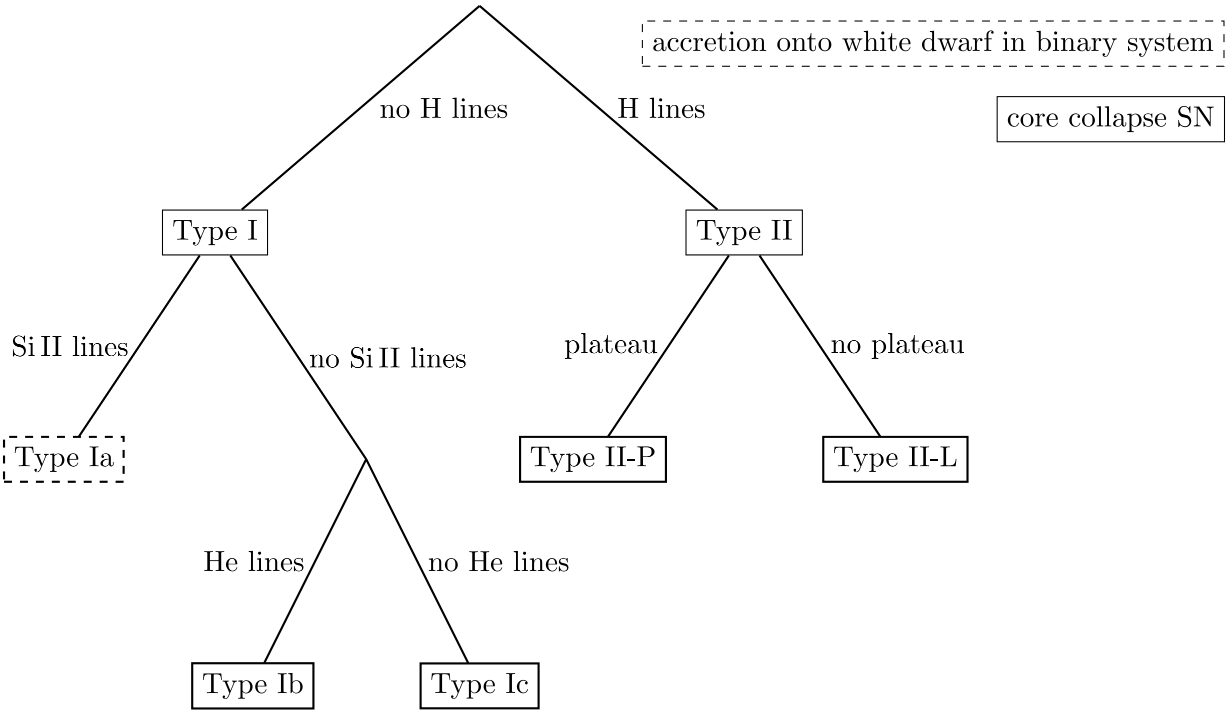


Figure 13.13:

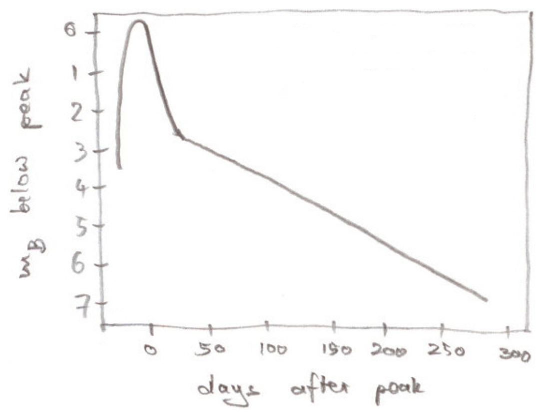


Figure 13.14: Type I SN (blue light curve)

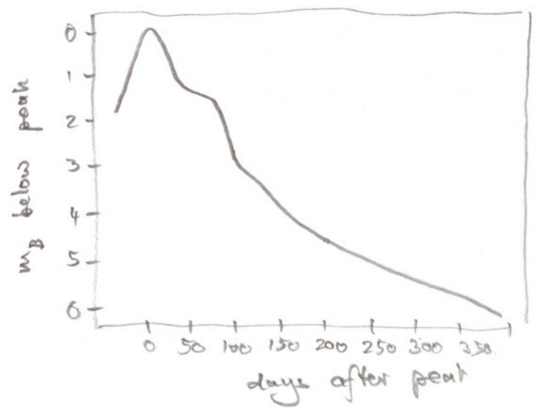


Figure 13.15: Type II-P SN (blue light curve)

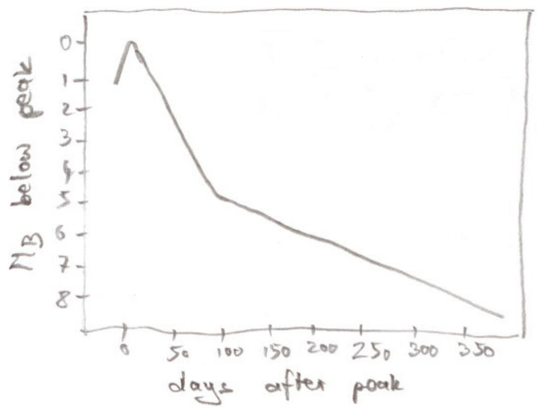


Figure 13.16: Type II-L SN (blue light curve)

- a typical Type II SN releases 10^{53} erg energy:

$\sim 99\%$	in neutrinos	
$\sim 1\%$	in kinetic energy of ejecta	similar values for Type Ib and Ic
$\sim 0.01\%$	in radiation	

- Type II, Ib and Ic are core collapse supernovae

  Type Ia occur in binary system, when mass transfer onto white dwarf causes ignition of degenerate He and sends a shock wave into CO core which ignites carbon and oxygen burning as well.
  The WD gets completely disrupted, leaving the binary companion "alone".

- core collapse SN:

  this type of SN occurs in shell-burning massive stars.
  He-shell adds C and O ashes to CO core, as the core continues to contract (not degenerate!) it ignites C burning, generating elements like

  $${}^{16}_{8}\mathrm{O},\ {}^{20}_{10}\mathrm{Ne},\ {}^{23}_{11}\mathrm{Na},\ {}^{23}_{12}\mathrm{Mg},\ {}^{24}_{12}\mathrm{Mg}$$

  this triggers a variety of further burning processes:
  Ne-O core will burn to Si.

  $$\begin{aligned} {}^{28}_{14}\mathrm{Si} + {}^{4}_{2}\mathrm{He} &\rightleftharpoons {}^{32}_{16}\mathrm{S} + \gamma \\ {}^{32}_{16}\mathrm{S} + {}^{4}_{2}\mathrm{He} &\rightleftharpoons {}^{36}_{18}\mathrm{Ar} + \gamma \\ {}^{52}_{24}\mathrm{Cr} + {}^{4}_{2}\mathrm{He} &\rightleftharpoons {}^{56}_{28}\mathrm{Ni} + \gamma \end{aligned}$$

  eventually an iron-rich core builds up

- at very high temperatures in the core **photo-disintegration** of heavy elements occurs:

  $$\begin{aligned} {}^{56}_{26}\mathrm{Fe} + \gamma &\longrightarrow 13\,{}^{4}_{2}\mathrm{He} + 4\mathrm{n} \\ {}^{4}_{2}\mathrm{He} + \gamma &\longrightarrow 2\mathrm{p}^{+} + 2\nu \end{aligned}$$

  $(T_c \approx 8 \cdot 10^9\,\mathrm{K}\ \&\ \varrho_c \approx 10^{10}\,\mathrm{g\,cm^{-3}})$

  now, the free electrons that had assisted in supporting the star against further collapse (by degeneracy pressure) are captured by protons:

  $$\mathrm{p}^{+} + \mathrm{e}^{-} \rightarrow \mathrm{n} + \nu_{\mathrm{e}}$$

  there is a gigantic burst of neutrino emission, the neutrino luminosity is orders of magnitude larger than nuclear burning luminosity.

  $\hookrightarrow$ degeneracy pressure is gone + enormous energy loss by ν_{e} emission.

  $\hookrightarrow$ **core collapse extremely rapidly.**

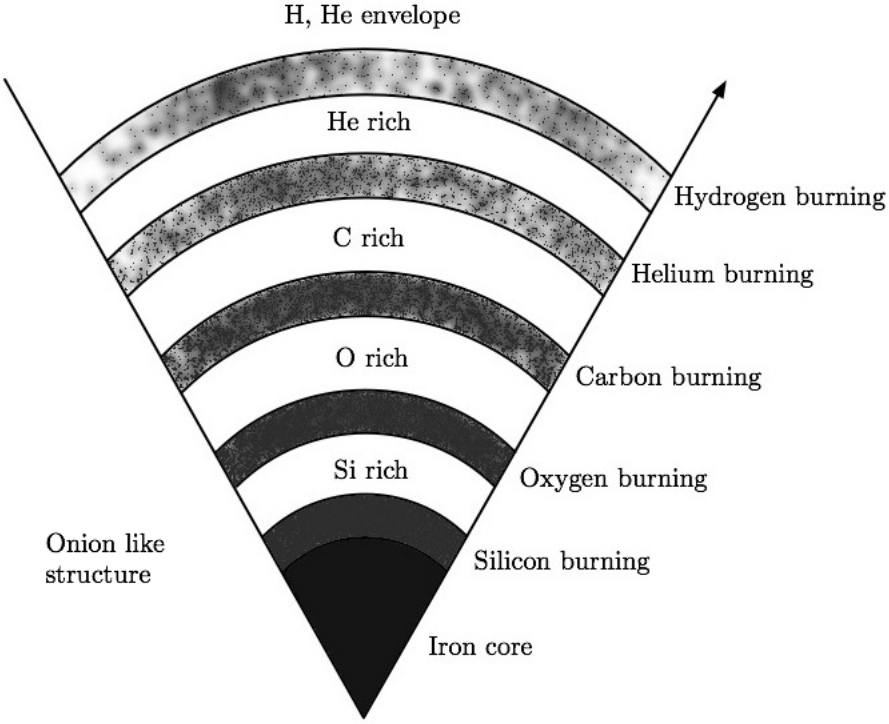


Figure 13.17:

depending on mass a **neutron star** remnant builds up in center
(for stars with $M_{ZAMS} \lesssim 25\,\mathrm{M}_\odot$)
$\hookrightarrow$ reverse shock expels outer layers
for stars with $M_{ZAMS} \gtrsim 25\,\mathrm{M}_\odot$ the remnant is too heavy and cannot be stabilized by n degeneracy $\rightarrow$ collapse to **black hole**!
still a reverse shock expels material just like with neutron star.

- the energy in the light curve comes from **radioactive decay** of expelled material:

$$
\begin{aligned}
{}^{56}_{28}\mathrm{Ni} &\rightarrow {}^{56}_{27}\mathrm{Co} + \mathrm{e}^+ + \nu_\mathrm{e} + \gamma \quad (\tau = 6.1\,\mathrm{days}) \\
{}^{56}_{27}\mathrm{Co} &\rightarrow {}^{56}_{26}\mathrm{Fe} + \mathrm{e}^+ + \nu_\mathrm{e} + \gamma \quad (\tau = 77.7\,\mathrm{days})
\end{aligned}
$$

- typical light curve:

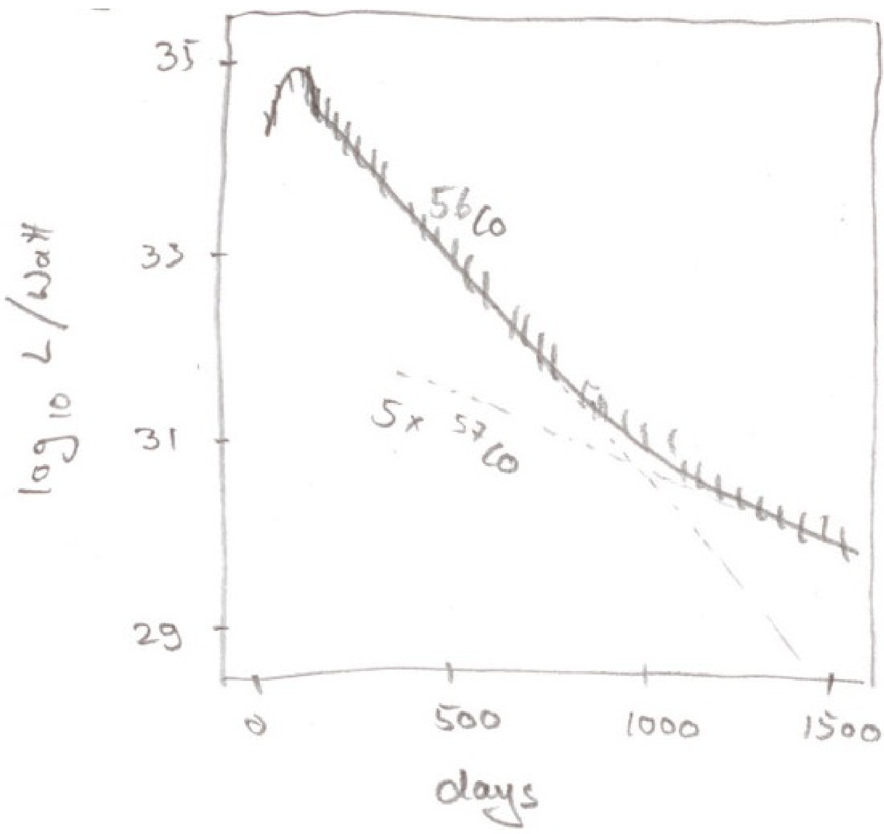


Figure 13.18:

- summary of post-main sequence evolution:

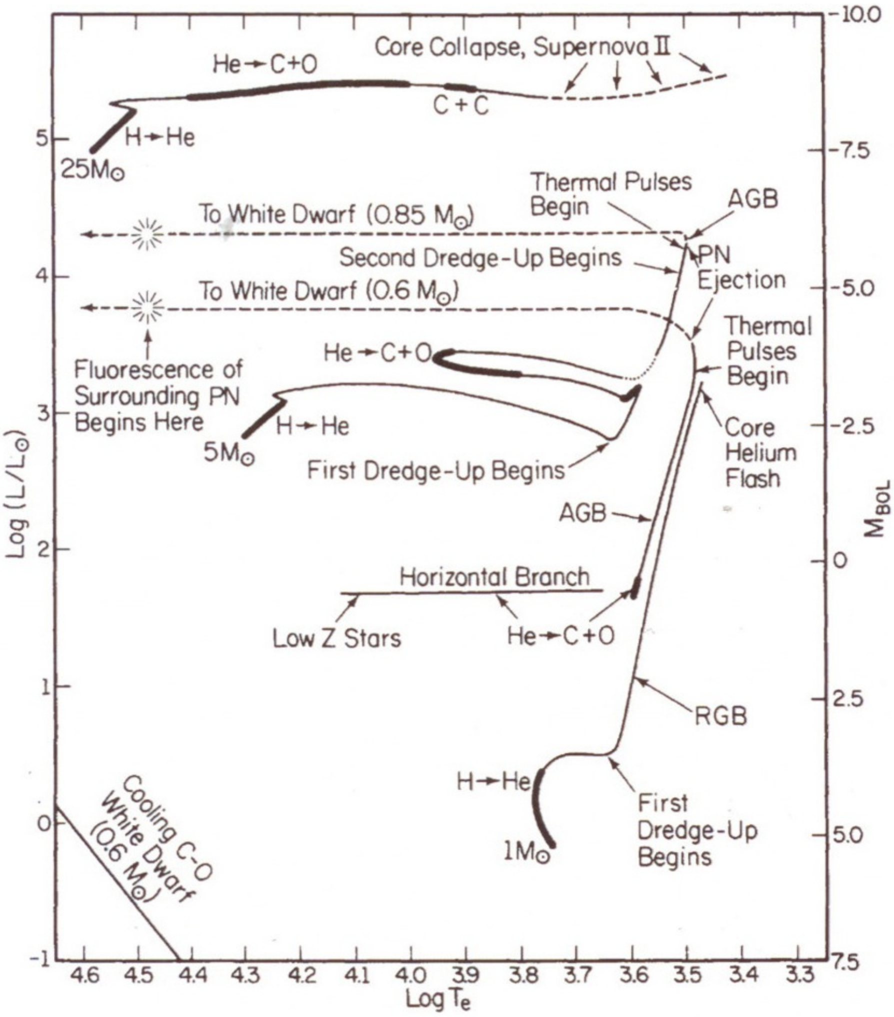


Figure 13.19: Tracks in the HR diagram of a representative selection of stars. [see Iben, 1985, Figure 15]

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