Stellar Astronomy and Astrophysics (SS12)

Stefan Jordan and Ralf Klessen

Exercise 6 for June 5, 2012

Simple models for stellar structure

6.1 Hertzsprung-Russell Diagram for Simple Model Stars

Consider a family of chemically homogeneous stars that are similar in every respect except for their masses M and radii R. Using a dimensionless radius variable, $x = r/R$, we can define similarity functions $F(x)$ such that

$$\rho(r) = \frac{M}{R^3} F_\rho(x) \qquad \text{and} \qquad m(r) = M\, F_m(x)\,.$$

Assume an ideal equation of state for the stellar material, i.e. $P = \rho k_\mathrm{B} T/\mu_m$ with Boltzmann's constant k_B and mean molecular mass $\mu_m = \mu\, m_\mathrm{p}$ (where μ =mean molecular weight and m_p =proton mass) and with P and T being pressure and temperature. Assume furthermore that energy is transported radiatively with an opacity obeying Kramer's law ($\kappa \propto \rho T^{-7/2}$) and that nuclear energy is generated by the PP chain where the energy production scales as $\varepsilon_\mathrm{PP} \propto \rho^2 T^4$.

a) Use the fundamental equations of stellar structure as outlined in the lecture to derive the following scaling relations for pressure P, temperature T, and energy flux due to radiatve transport L_rad as well as due to nuclear fusion L_nuc:

$$\begin{aligned} P(r) &= \frac{M^2}{R^4} F_P(x)\,, \\ T(r) &= \frac{M}{R} F_T(x)\,, \\ L_\mathrm{rad}(r) &= \frac{M^{5.5}}{R^{0.5}} F_\mathrm{rad}(x)\,, \\ L_\mathrm{nuc}(r) &= \frac{M^6}{R^7} F_\mathrm{nuc}(x)\,, \end{aligned}$$

where again the F-functions are common to all family members.

b) Note that the energy flux transported by radiative diffusion increases slowly while the flux generated by nuclear fusion rises rapidly as the star contracts. Sketch L_rad and L_nuc as function of radius. Find the radius and luminosity as function of total mass at which the PP chain can produce enough energy to compensate the radiative losses at the surface. This is when the star reaches a quasi-equilibrium state: the stellar main sequence.

c) Demonstrate that all stars of the homologous family in this phase lie on a line in the Hertzsprung-Russell diagram with

$$L \propto T_\mathrm{E}^{4.12}$$

with T_E being the effective temperature at the surface.

6.2 Polytropic spheres:

The Lane-Emden equation for a polytropic index n ($P = K\rho^\gamma = K\rho^{(n+1)/n}$) is

$$\frac{1}{\xi^2}\frac{\mathrm{d}}{\mathrm{d}\xi}\left[\xi^2\frac{\mathrm{d}D_n}{\mathrm{d}\xi}\right] = -D_n^n\,,$$

where $\rho(r) = \rho_c[D_n(r)]^n$ with $0 \le D_n \le 1$, and where $r = \lambda_n\xi$ with $\lambda_n = \left[(n+1)\left(\frac{K\rho_c^{(1-n)/n}}{4\pi G}\right)\right]^{1/2}$. The surface is defined by $D_n(\xi_{\rm surf}) = 0$, and the inner boundary is $\mathrm{d}D_n/\mathrm{d}\xi = 0$ at $\xi = 0$. Usually, solutions can only be obtained numerically. However, there are three values of n for which analytic expressions can be obtained.

(a) Show that the $n = 0$ polytrope has a solution given by

$$D_0(\xi) = 1 - \frac{\xi^2}{6}, \text{ with } \xi_{\rm surf} = \sqrt{6}.$$

(b) Show that for $n = 1$ the solution is

$$D_1(\xi) = \frac{\sin\xi}{\xi}, \text{ with } \xi_{\rm surf} = \pi.$$

(c) There is also an analytic solution for $n = 5$. It is given by

$$D_5(\xi) = (1 + \xi^2/3)^{-1/2}, \text{ with } \xi_{\rm surf} \to \infty.$$

Is the total mass finite?

(d) Plot the density structure for $n = 0, 1,$ and 5 (ρ_n/ρ_c vs. r/λ_n).

This exercise is voluntary.