Homework Assignment #7 is due Wednesday, Dec. 2, 2015

Theoretical Astrophysics (MKTP2)

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Ralf Klessen, ZAH/ITA, Albert-Ueberle-Str. 2, 69120 Heidelberg

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1. **Stability of magnetized gas clouds** *15 pt*

   Consider a spherical, self-gravitating gas cloud with homogeneous density which is threaded by a uniform magnetic field of strength $\vec{B}$. Assume that the cloud has a radius of $r = 0.1$ pc and a mean density of $\bar{\rho}_n = 1.67 \times 10^{-20}$ g cm^{-3}.

   (a) Calculate the mass-to-flux ratio given by $\mu = M_{\rm cl}/\Phi$, where $M_{\rm cl}$ is the total mass of the cloud and $\Phi = \pi r^2 |\vec{B}|$ is the total magnetic flux in the scenario under consideration. Assume a magnetic field strength of $B = 30\mu$G.

   (b) The critical mass-to-flux ratio for gravitational collapse of the magnetized cloud can be derived as $\mu_{\rm crit} = 0.13 G^{-1/2}$, where G is the gravitational constant. Calculate the critical magnetic field strength $B_{\rm crit}$ for which the cloud is stable against gravitational collapse.

   (c) The ambipolar diffusion timescale is defined as

   $$\tau_A = 4\pi\gamma\rho_i\rho_n L^2/B^2 \tag{1}$$

   where ρ_i and ρ_n denote the density of the ions and neutrals in the gas, respectively, and L corresponds to the size of the cloud. Assuming that ρ_i is given by $3.3 \times 10^{-27}\,\mathrm{g\,cm^{-3}} \cdot \left(\rho_n/[3.85 \times 10^{-19}\,\mathrm{g\,cm^{-3}}]\right)^{1/2}$ and the friction coefficient is $\gamma = 3.3 \times 10^{13}$ g^{-1} cm^3 s^{-1}, calculate τ_A for the magnetic cloud considered above.

   Note that in steady state, the rate of recombinations of electrons and ions is proportional to ρ_i^2 and that it equals the rate at which cosmic rays ionize neutrals, which is proportional to ρ_n. Therefore we get $\rho_i \propto \rho_n^{1/2}$, explaining the above relation.

   Also calculate the corresponding gravitational free-fall timescale $\tau_{\rm ff} = (3\pi/32 \mathrm{G}\rho_n)^{1/2}$ of the system. Discuss the consequences of the difference between the two timescales. What do you conclude about the evolution of the gas cloud?

2. **Magnetic Braking of an Alligned Rotor** *25 (+ 10) pt*

Consider a uniform gaseous disk of density $\rho_{\rm cl}$ and half-thickness Z rotating rigidly with an initial angular velocity Ω_0. Furthermore assume the disk is threaded with a magnetic field $\vec{B}$ of strength B_0, initially uniform and parallel to the rotation axis. The magnetic field links the disk with the external medium of density $\rho_{\rm ext}$ which is initially at rest. Assume axisymmetry and use cylindrical coordinates (R, φ, z) for the calculations.

(a) Derive the evolution equations for the toroidal magnetic field B_φ and the angular velocity Ω (where $v_\varphi = R\,\Omega$) outside of the disk, i.e. $|z| > Z$, assuming that the radial velocity v_r and the poloidal velocity v_z are negligible small (compared to the Alfvén velocity).

(b) Show that the evolution of the external medium can be expressed by the wave equation

$$\frac{\partial^2 \Omega}{\partial t^2} = v_{\rm A,ext}^2 \, \frac{\partial^2 \Omega}{\partial z^2} \tag{2}$$

where $v_{\rm A,ext} = B_0/\sqrt{4\pi\,\rho_{\rm ext}}$ is the Alfvén velocity in this medium.

(c) Derive the evolution equation of the angular velocity at the surface of the disk, i.e. $|z| = Z$, using the torque per unit area, $N = R\,B_0\,B_\varphi/4\pi$, which the magnetic field exerts on the surface of the disk. Result:

$$\frac{\partial^2 \Omega_{\rm cl}}{\partial t^2} = \frac{1}{Z}\,\frac{\rho_{\rm ext}}{\rho_{\rm cl}}\,v_{\rm A,ext}^2 \left.\frac{\partial \Omega}{\partial z}\right|_{|z|=Z} \tag{3}$$

(d) *Bonus:* Combine equations (2) and (3) to calculate the spin-down time of the disk. (+10 pt)
Hint: Use the solution of equation (2) at the disk surface $|z| = Z$.