Continuity Equation

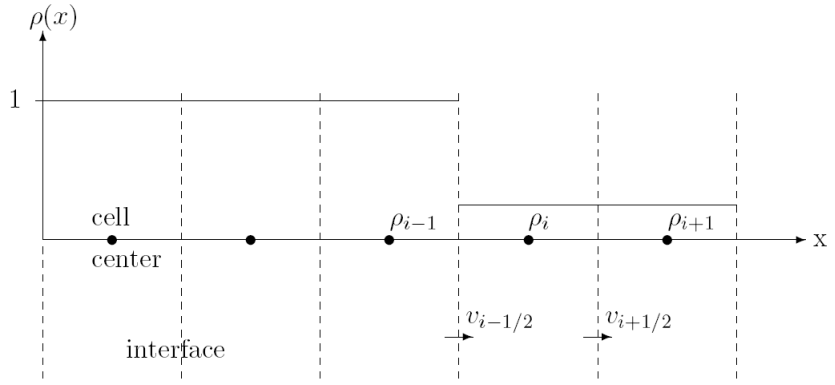
Problem sheet 4

19/05/2009

Doner-Cell : a conservative scheme

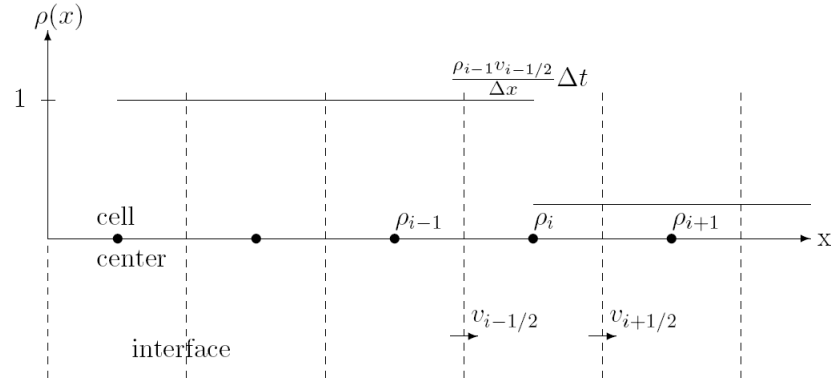
In problem sheet 2 we achieve a stable discretization scheme independent of the sign of the constant (space-independent) velocity v_0. In the extra exercise is was shown that this "Upwind" method is not conservative in the case of a space-dependent velocity distribution $v(x)$.

Now we want to achieve a conservative scheme (called "Donor-Cell") in describing the expression $\rho(x,t)v(x)$ as the density flux over the interfaces between the specified grid points. Therefore we want to 'store' the density value (integrated distribution over the cell) at the cell centers and the velocity at the interface (a so called staggered mesh):



1. Begin with the 1D continuity equation of hydrohynamics.
2. Substitute the expression for the density flux with $F(x,t) = \rho(x,t)v(x,t)$
3. Discritize the time deriative in first order approximation as usual.
4. The flux will be stored at the cell interfaces (because it is a vector quantity like the velocity itself). Discretize the fluxes (into or out of the cell i) in a symmetric explicit method: $\partial_x F(x,t) \to \frac{F^n_{i+1/2} - F^n_{i-1/2}}{\Delta x}$. That means the

change of the density in the i'th cell is calculated from the density flux over the left and right interface of the cell:



With a positive velocity $v(x) = 1$, a grid spacing of $\Delta x = 1$ and a timestep $\Delta t = 0.5$ this picture would describe the analytic solution of the given distribution above after the time Δt. Each flux which is added to one cell is substracted from another cell, that means, this scheme conserves the density.

5. At each timestep we want to calculate these fluxes $F^n_{i-1/2}$ as the product of the velocity $v_{i-1/2}$ and the density from the corresponding Upwind scheme (ρ_{i-1} for positive and ρ_i for negative velocity at $i-1/2$). Calculate the final expression for the new density of the i'th cell ρ_i^{n+1}. The flux (for example at the left interface) in Upwind condition can be written and later on implemented pratically as $F_{i-1/2} = \rho_{i-1}\max(v_{i-1/2}, 0) + \rho_i \min(v_{i-1/2}, 0)$.

6. What would you get, if you would now simplify the problem by assuming a space-independent velocity $v_0 = \pm 1$? Compare the resulting expression with the discretization schemes of problem sheet 2.

7. Program this scheme and test it with the initial density and velocity distribution from the problem sheet 2
   Hints: Of course you have to store the velocities and fluxes at a specific interger (not at $i \pm 1/2$). Add therefore the value of $1/2$ to the space subscript of all vector quantities: $v_{i-1/2} \rightarrow v_i$ etc.

Code guidance: In each timestep...

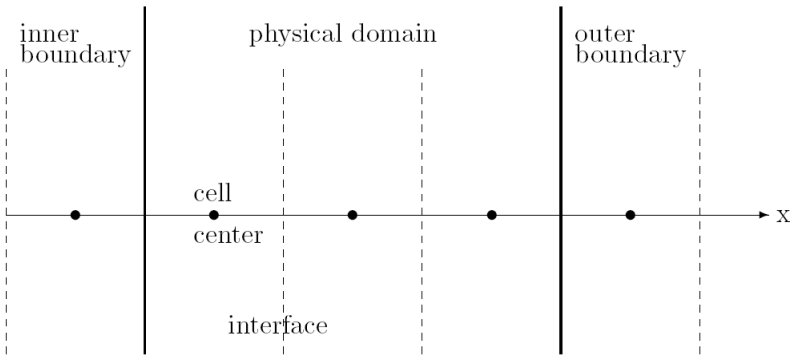
- calculate first the density fluxes at the interfaces using the Upwind scheme F_i
- update afterwards the density distribution in conservative format $\rho_i^{n+1} =$ .....
- in the end set the values for the boundary cell equals zero: $\rho_0^{n+1} = \rho_{imax}^{n+1} = 0$.

Boundary Condition

Introduction:

Up to now we avoid such large time integrations, that the initial step function reaches the border of the computational domain (beginning or end of the x-axis). To describe the physics at this border correctly (for instance if the beginning of the axis describes the solid earth surface and the end of the axis reaches 10 km upwards into the atmosphere), we have to establish different conditions at the borders.

Furthermore it is useful to add some so called boundary, ghost or halo cells to the computational domain, to use the same stencil (e.g. $\frac{\rho_{i-1}-\rho_i}{\Delta x}$) in looping over the physical domain (we have to add as many ghosts as our stencil is using , at the moment only one at the each side, but this will change during the following weeks):



Implement different boundary condition into the prior conservative advection code. When the step function hits one of the computational borders, the density

1. is assumed to shrink to zero (Dirichlet)
2. is assumed to be constant over the boundary (Zero Gradient)
3. should enter the domain at the opposite side again (the axis describes a curved space, circle, i.e., periodic)