I Retrospect and working plan

Goal: Gauge-invariant perturbation variables
- clear physical and geometrical meaning
- no gauge modes (Ellis & Bruni 1985)

Tool: Harmonical analysis, which means a decomposition into eigenfunctions of the Laplacian
- Allows to treat perturbation different kinds of perturbations separately
  - $V_i = V \phi_i^{(1)} + V^{(2)} \phi_i^{(2)}$ - depend on geometry
  - $H_{ij} = H_e \phi_i^{(1)} \phi_j^{(1)} + H_i \phi_i^{(1)} + H^{(2)} \phi_i^{(2)} + H^{(3)} \phi_i^{(3)}$
  (all coefficients are functions of $\Phi$ and $t$)

How to:
1. Look at first-order metric perturbations
   $h_{\mu \nu} \, dx^\mu \, dx^\nu = -2 \Delta dt^2 - 2 B_{ij} \, dx^i \, dx^j$ and work out their behaviour under gauge transforms.
2. Look at first-order energy-momentum tensor perturbations.
3. Find perturbation equations using Einstein's equation.
4. Try to solve them.
Permutations of the energy-momentum tensor

The perturbed tensor reads:

\[ T^\nu_{\ \nu} = \tilde{T}^\nu_{\ \nu} + \Theta^\nu_{\ \nu} \]

We define energy density \( \bar{s} \) and energy flux as:

\[ T^\nu_{\ \mu} \ u^\mu = - \bar{s} \ u^0 \quad \text{and} \quad u^2 = -1 \]

and parametrise their perturbations

\[ \delta = \bar{s} (1 + \delta) \quad \text{and} \quad u = u^0 \mathcal{E} + u^i \mathcal{E}_i \]

Stress tensor

Define a projection onto the subspace of tangent space normal to \( u \):

\[ p^\nu_{\ \nu} = m^\nu_{\ \nu} + \delta^\nu_{\ \nu} \]

And the stress tensor:

\[ \tilde{\tau}^\nu_{\ \nu} = p^\alpha_{\ \nu} p^\beta_{\ \nu} T^\alpha_{\ \beta} \]

This allows us to write:

\[ T^\nu_{\ \nu} = \bar{s} m^\nu_{\ \nu} + \tilde{\tau}^\nu_{\ \nu} \]

because:

\[ \bar{s} m^\nu_{\ \nu} + \tilde{\tau}^\nu_{\ \nu} = \bar{s} m^\nu_{\ \nu} + p^\alpha_{\ \nu} p^\beta_{\ \nu} T^\alpha_{\ \beta} \]

\[ = \bar{s} m^\nu_{\ \nu} + (m^\nu_{\ \nu} u_\alpha + \delta^\nu_{\ \alpha})(m^\nu_{\ \nu} u_\alpha + \delta^\nu_{\ \alpha}) T^\alpha_{\ \beta} \]

\[ = \bar{s} m^\nu_{\ \nu} + m^\nu_{\ \nu} m_\alpha u^\alpha m^\nu_{\ \nu} T^\alpha_{\ \beta} + m^\nu_{\ \nu} \delta^\nu_{\ \alpha} T^\alpha_{\ \beta} + m^\nu_{\ \nu} \delta^\nu_{\ \alpha} T^\alpha_{\ \beta} \]

\[ + \delta^\nu_{\ \alpha} \delta^\nu_{\ \beta} T^\alpha_{\ \beta} \]
\[\begin{align*}
&= g^{u\nu} m_v m_u + \left(\frac{\gamma}{\mu} - \frac{1}{\mu} v^2\right) m^2 m_v m_u^2 + m\mu m_v T^u_{\nu} + T^u_{\nu} \\
&= g^{u\nu} m_v m_u + \left(\frac{\gamma}{\mu} - \frac{1}{\mu} v^2\right) m^2 m_v m_u^2 - g^{u\nu} m_v m_u^2 + T^u_{\nu}
\end{align*}\]

In the homogeneous case we have:

\[c^0 \equiv c^{\alpha}_\alpha = 0 \quad \Rightarrow \quad c^\alpha = \mp \delta^\alpha\]

If we define velocity components:

\[u^i = \frac{1}{a} v^i = \frac{1}{a} \left( v^{(\nu)} q^{(\nu) i} + v^{(\nu)} q^{(\nu) i} \right)\]

we find with the metric Petrov-Fierz:

\[C^\alpha_{\beta} = 0 \quad \Rightarrow \quad c^0 = -\bar{p} v^i; \quad \bar{c}^\alpha = \bar{p} (v^3 - h^3)\]

For \(c^\alpha\) we have to introduce new Petrov-Fierz:

\[c^\alpha = \bar{p} \left( (\gamma + \bar{p}) \delta^\alpha \right) + \bar{\Pi}^\alpha \]

\[\bar{\Pi}^\alpha = 0\]

\[\text{anisotropic stress}\]

\[\bar{\Pi}^\alpha = \bar{\Pi}^{(\nu)} q^{(\nu) i} + \bar{\Pi}^{(\nu)} q^{(\nu) i} + \bar{\Pi}^{(\nu)} q^{(\nu) i}\]

Then we find:

\[T^u_{\nu} = \begin{pmatrix}
-\bar{p}(1 + \delta) & (\bar{p} + \bar{p}) (v^2 - h^2) \\
-(\bar{p} + \bar{p}) v^2 & \bar{p} \left( (\gamma + \bar{p}) \delta^\alpha + \bar{\Pi}^\alpha \right)
\end{pmatrix}\]
\[ S^{(n)} \rightarrow S^{(n)} + L_x \bar{S} \]

for \( S = \bar{S} + \epsilon S^{(n)} \) for arbitrary quantities, we find:

\[ S \rightarrow \bar{S} - 3(1+w) \epsilon T \]
\[ \bar{c} \rightarrow \bar{c} - 3 \frac{\epsilon^2}{w} (1+w) \epsilon T \]
\[ \bar{v} \rightarrow \bar{v} - \bar{z} \]
\[ \bar{\Pi} \rightarrow \bar{\Pi} \]
\[ \bar{V}^{(\nu)} \rightarrow \bar{V}^{(\nu)} - \bar{Q}^{(\nu)} \]
\[ \bar{\Pi}^{(\nu)} \rightarrow \bar{\Pi}^{(\nu)} \]
\[ \bar{\Pi}^{(1)} \rightarrow \bar{\Pi}^{(1)} \]

Out of this, you can find only one gauge-invariant perturbation variable:

\[ \Gamma = \bar{c} - \frac{\epsilon^2}{w} S \]

To find others, we need the metric perturbation:

\[ V = V - \frac{1}{\kappa} \bar{H} = V_{\text{new}} \]
\[ D_3 = \frac{1}{\kappa^2} \kappa \bar{\Omega} (8^2 - \bar{H}^2 - 8 \bar{\gamma} B) = \delta_{5,5} \]
\[ D = \delta_{5,5} + 3(1+w) \frac{\kappa}{\kappa^2} V = S + 3(1+w) \frac{\kappa}{\kappa^2} (V - B) \]
\[ D_3 = S + 3(1+w) \left( \frac{1}{\kappa} \bar{H}_c + \frac{2}{3} \bar{H}_T \right) = \delta_{5,5} - 3(1+w) \phi \]
\[ V^{(\nu)} = V^{(\nu)} - \frac{1}{\kappa} \bar{H}^{(\nu)} = V_{\text{new}} \]
\[ \Omega = V^{(\nu)} - B^{(\nu)} = V_{\text{new}} - B^{(\nu)} \]
\[ \Omega = V^{(\nu)} - B^{(\nu)} \quad \text{(not just perturbations in vector sense)} \]
we identified all gauge-invariant variables and have to derive their EOM's.

Two "standard" ways:

1.) Go through the Christoffel in a certain sense (e.g. Doolson does it similarly)

2.) Use 3+1 formulation of GR and Carter's formalism (Durrer and Straumann 1988)

we find:

\[ G_{\alpha\mu} = 8\pi G T_{\alpha\mu} \quad \text{(constraints)} \]

\[
4\pi G a^3 \phi' = -(x^2 - 3K) \phi \quad \text{scalar}
\]

\[
4\pi G a^3 (g + p) \nabla = x (\mathcal{H} \Gamma + \phi) \quad \text{scalar}
\]

\[
8\pi G a^3 (g + p) \Omega = \frac{1}{2} (2K - x^2) \delta^{(s)} \quad \text{vector}
\]

\[
G_{ij} = 8\pi G T_{ij} \quad \text{(dynamical)}
\]

\[
x^2 (\phi - \nabla) = 8\pi G a^3 P \Pi^{(s)} \quad \text{scalar}
\]

\[
\dot{\phi} + 2\mathcal{H} \phi + \mathcal{H} \dot{\Gamma} + [2 \mathcal{H} + x^2 - \frac{2}{3}] \Gamma = 4\pi G a^2 \phi' \left[ \frac{3}{2} \Gamma + c_s^2 \nabla + \omega \Gamma \right] \quad \text{scalar}
\]

\[
x (\delta^{(s)} + 2 \mathcal{H} \delta^{(s)}) = 8\pi G a^3 P \Pi^{(s)} \quad \text{vector}
\]

\[
\Pi^{(s)} + 2\mathcal{H} \Pi^{(s)} + (2K + x^2) \Pi^{(s)} = 8\pi G a^2 \mathcal{P} \Pi^{(s)} \quad \text{tensor}
\]

\[
w = \frac{p}{\rho} \quad \text{c}_s = \frac{p'}{\rho} = \frac{\dot{\rho}}{\rho}
\]
Some remarks:

- For a perfect fluid we have $\Pi^2 = 0$
  
  $\Rightarrow \phi = 1$

- Adiabatic perturbations are characterised by $\Gamma = 0$

- Scalar perturbations with $\Pi = \Gamma = 0$ are described by a damped wave equation with the propagation speed $c_s^2$.

- $H^{(1)}$ in perfect fluids is also given by a damped wave equation, where the damping can be neglected on small scales and short time periods.
  
  $\Rightarrow$ Gravitational wave propagation with the speed of light.

- Vector perturbations simply decay.

Energy - momentum conservation

$T_{\mu\nu} = 0$

\[
0 + 3(c_s^2 - w) \partial \phi D_y + (1 + w) \partial \nu + 3w \partial \Gamma = 0
\]

\[
\partial \imath (1 - 3c_s^2) \nu = 8(1 + 3c_s^2 \phi) + \frac{c_s^2 \partial \nu}{\Delta^2} + \frac{\nu}{7 + w} \left[ \Gamma - \frac{2}{3} (1 - \frac{K}{\Delta^2}) \right]
\]

\[
\partial + (1 - 3c_s^2) \partial \phi \Delta = -\frac{w}{2(1 + w)} (\Delta - \frac{2K}{\Delta}) \Pi^{(v)}
\]
Some more remarks

- Scalar perturbations: 4 independent equations
  6 perturbation variables

- Vector perturbations: 2 equations
  3 variables

- Tensor perturbations: 1 equation
  2 variables

One way to close the system are adiabatic perturbations of the perfect fluid:

\[ \Gamma = \Pi_{ij} = 0 \]

The conservation

- Different fluid components:
  - The conservation equations hold for each component separately
  - In the Einstein equation one has to be careful, since the metric perturbations are induced by the full perturbation:

\[
\begin{align*}
\delta g &= \delta \alpha \delta g_{\alpha x} \\
(\delta + \delta) V &= (\delta \alpha + \alpha \delta) V_{\alpha x} \\
p \Pi &= p_{\alpha} \Pi_{\alpha x} \\
p \Gamma &= p_{\alpha} \Gamma_{\alpha x} + p \Gamma_{\text{rel}}
\end{align*}
\]
The Bardeen equation

For late use we want a scalar perturbation equation for the source Bardeen potentials with \( \Gamma \) and \( \Pi \) as sources:

**Recipe:**
1. Take 2nd scalar Einstein equation
2. Replace \( D_s \) by \( D \)
   \[
   0 = D_s + 3(\psi + \omega) \frac{\partial}{\partial \nu}
   \]
3. Replace \( D \) via "Poisson equation."
4. Take 1st scalar equation and differentiate
5. Replace \( \dot{x} \) and \( \ddot{x} \)
6. Use \( \dot{\phi} = -3(3 + \rho) \dot{x} \)
   \[
   \dot{x} = \frac{a^2}{\Lambda} \frac{\ddot{x}}{x} = - \frac{1 + 3\omega}{2} \left( \dot{x}^2 + K \right)
   \]

This results in:

\[
\ddot{\phi} + 3 \dot{x} \left( 1 + c_s^2 \right) \dot{\phi} + \left[ 3 c_s^2 - \omega \right] \dot{x}^2 - \left( 2 + 3\omega + 3c_s^2 \right) K + c_s^2 \dot{\phi} = \frac{8\pi G a^2 p}{\Lambda^2} \left[ \dot{x} \dot{\Pi} + \left[ 2 \dot{x} \dot{x} + 3 \dot{x}^2 \left( 1 - \frac{c_s^2}{\omega} \right) \right] \Pi - \frac{3 \dot{x}^2}{2} \Pi + \frac{\dot{x}}{2} \right]
\]
we discuss adiabatic scalar perturbations in a perfect fluid.

\[ \Pi = \Gamma = 0 \]

\[ \frac{\Delta}{\phi - \Pi} - 8 \pi G a^2 \rho \Pi (\phi) \]

\[ \Rightarrow \phi - \Pi \]

with Bardeen equation:

\[ \ddot{\phi} + 3 \mathcal{H} (1 + c_s^2) \dot{\phi} + \mathcal{E} (1 + 6 c_s^2) \dot{\phi} = K \]

\[ \ddot{\phi} + 3 \mathcal{H} (1 + c_s^2) \dot{\phi} + \left[ (1 + 6 c_s^2) (\mathcal{E} - K) - (1 + 3 w) (\mathcal{E} + K) + c_s^2 k^2 \right] \phi = 0 \]

This is a damped wave equation. Furthermore we neglect curvature and assume \( w = \text{const} \).

\[ \Rightarrow K = 0 \quad \text{and} \quad w = c_s^2 \]

\[ \Rightarrow \ddot{\phi} + 3 \mathcal{H} (1 + c_s^2) \dot{\phi} + w^2 \phi = 0 \]

Now we use:

\[ a \alpha + \frac{2}{1 + 3 w} = 1 \]

\[ \Rightarrow \mathcal{H} = \frac{\dot{a}}{a} = \frac{2}{1 + 3 w} \frac{1}{t} \]

\[ q = \frac{2}{1 + 3 w} \]
This has an analytic solution:

\[ n^T = \frac{1}{a} \left( A_J \left( \sqrt{\omega} \eta T \right) + B_Y \left( \sqrt{\omega} \eta T \right) \right) \]

with the spherical Bessel functions of order \( q \).

We distinguish between sub and super-horizon scales:

\[
\begin{align*}
  n^T &= \begin{cases} 
    \text{constant} & \text{for } \sqrt{\omega} \eta T \ll 1 \\
    \frac{A}{a \sqrt{\omega} \eta T} \sin(\sqrt{\omega} \eta T) - \frac{q}{2 \pi} & \text{for } \sqrt{\omega} \eta T \gg 1 
  \end{cases}
\end{align*}
\]

Assuming:

- radiation domination in the beginning
- matter domination at \( t_{eq} \)
- scale invariant initial spectrum

\( \Rightarrow \) Power spectrum in later (matter-dominated) times:

\[
\langle 1 + l^2 \rangle \propto \mathcal{P}_s = A_s \left( \frac{\mathcal{R}}{H_0} \right)^{n_s - 1} \begin{cases} 
    1 & \text{for } \mathcal{R}_{eq} < 1 \\
    \left( \mathcal{R}_{eq} \right)^{-4} \cos^2 \left( \mathcal{R}_{eq} \right) & \text{for } \mathcal{R}_{eq} > 1
  \end{cases}
\]