1 Introduction to lensing

1.1 Light deflection and resulting phenomena

It had been speculated even by Newton that masses should deflect light, but he did not know how to describe the deflection properly, because he thought of light as only a wave phenomenon.

In 1783, speculating that light consists of corpuscoles, a geologists, astronomer, natural phylosopher and what-so-ever, named John Mitchell (1724-1793) sent to Henry Cavendish (1731-1810) a paper he had written on a method to measure the mass of stars by detecting the reduction in the light speed by effect of gravity as the light corpuscoles propagated from the star's gravitational field to the Earth. Among the other things, in this paper Mitchel suggested that a sufficiently massive body could completely stop the light it emitted and appear as invisible (hey, aren't these balck holes?).

The paper from Mitchell pushed Cavendish to calculate the Newtonian deflection of light for the first time, probably around 1784. Unfortunately, he did not pubilsh his results. Some private notes where discovered only later.

The calculation was as follows (Will, 1988):

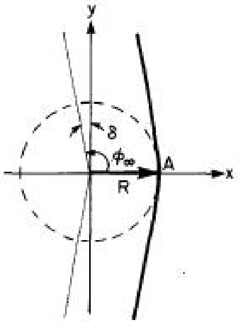
- let start from the assumption that light is composed of material corpuscles;
- according to the equivalence principle, the acceleration of a body in a gravitational field is independent of its mass, structure, composition. Therefore we do not need to care about the corpuscle mass;
- any light corpuscle should experience the acceleration

  $$\frac{\mathrm{d}^2\vec{r}}{\mathrm{d}t^2} = -\frac{Gm\vec{r}}{r^3}\,, \tag{1.1}$$

  where $\vec{r}$ define the position of the corpuscle in the gravitational field of the body whose mass is m;
- the solutions of this equation of motion are conic sections. They can describe bound or unbound orbits. However, the speed of light is so large that it exceeds the escape velocity. Thus, the resulting orbit will be an hyperbolic orbit, which can be parametrically written as

  $$r = \frac{R(1+e)}{1+e\cos\phi} \quad , \quad r^2\frac{\mathrm{d}\phi}{\mathrm{d}t} = [GmR(1+e)]^{1/2}\,, \tag{1.2}$$

In the previous equations R is the radius of the point of closest approach between the corpuscle and the body of mass m, chosen to lie of the x axis, e is the eccentricity of the orbit and ϕ is an angle, counted from the x axis, called *true anomaly*. r and ϕ define the position of the corpuscle with respect to the mass m in polar coordinates.



- the vecor $\vec{r}$ is written as

$$\vec{r} = r(\vec{e}_x \cos\phi + \vec{e}_y \sin\phi) \tag{1.3}$$

in terms of the two components along the x and the y axes. Thus, the velocity $\vec{v}$ is

$$\begin{aligned} \vec{v} &= \frac{\mathrm{d}\vec{r}}{\mathrm{d}t} = \left(\frac{Gm}{R(1+e)}\right)^{1/2} \left[-\vec{e}_x \sin\phi + \vec{e}_y(\cos\phi + e)\right] , & (1.4) \\ v^2 &= \frac{Gm}{R(1+e)}(1 + 2e\cos\phi + e^2) . & (1.5) \end{aligned}$$

- as $r \to \infty$, the trajectory approaches asymptotes that make an angle ϕ_∞ with the x-axis; this occurs when

$$(1 + e\cos\phi) = 0 \Rightarrow \cos\phi_\infty = -\frac{1}{e} . \tag{1.6}$$

If we define $\phi_\infty \equiv \pi/2 + \delta$, where δ is one-half the deflection angle, then

$$\sin\delta = \frac{1}{e} ; \tag{1.7}$$

- for determining the deflection angle, we need to determine the eccentricity. Now, let assume that the corpuscle is emitted at infinity with velocity c. Then, from Eq. 1.5 we obtain

$$\begin{aligned} c^2 &= v^2|_{\phi=\phi_\infty} = \frac{Gm}{R(1+e)}(e^2 - 1) & (1.8) \\ &= \frac{Gm}{R}(e - 1) . & (1.9) \end{aligned}$$

Thus,

$$e = \frac{Rc^2}{Gm} + 1 ; \tag{1.10}$$

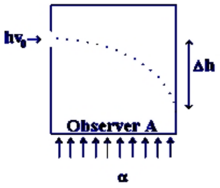
- if the massive body is the Sun and the light is grazing its surface,

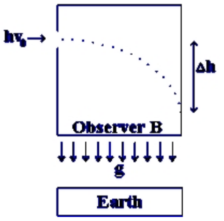
$$\begin{aligned} m &= M_\odot = 1.989 \times 10^{30}\mathrm{kg} & (1.11) \\ R &= R_\odot = 6.96 \times 10^{8}\mathrm{m} & (1.12) \end{aligned}$$

and the deflection angles is

$$\Delta\theta \equiv 2\delta \approx \frac{2Gm}{c^2 R} \approx 0''.875 \tag{1.13}$$

However, we have to wait until the beginning of the XIXth century for finding an official document by Johann Soldner (1801), where these calculations were published. The result shown above is just one half of the true deflection, because it is derived by neglecting the local curvature of the space-time around massive bodies.





Using an argument based on the principle of equivalence, but still without full equations of relativity, Albert Einstein realized that massive bodies deflect light. The argument works like this. The principle of equivalence states that gravity and acceleration cannot be distinguished. In other words, a free falling observer does not feel gravity and an accelerated observer can interpret the resulting inertial force as due to a gravitational field. Suppose that the observer is contained in a box with a hole on its left side (see upper figure). If the box is accelerated upwards, the observer interprets the inertial force on him as a gravitational force acting downwards. Suppose that a light ray enters the hole on the left side of the box and propagates towards right. As the box is moving upwards, the ray hits the wall of the box on the opposite side at a lower point than it enter. As the box is accelerated the light ray appears curved. Then, based on the principle of equivalence, light must be deflected by gravity. Indeed, we can imagine to reverse the experiment: let the box to be stationary and within the gravitational field whose intesity is such to resemble the previous acceleration. If light is not deflected by gravity, then the oberver has the possibility to discriminate between gravity and acceleration, violating the principle of equivalence.

In order to get the correct value of the deflection of light by a mass M, we need to use the Theory of General Relativity (Einstein, 1916). According to this theory, the deflection is described by geodesic lines following the curvature of the space-time. In curved space-time, geodesic lines are lines which are as "straight as possible", resembling straight lines in flat space-time. As a light ray follows the curvature, it is bent towards the mass which causes the space-time to be curved. This bending gives rise to several important phenomena:

- multiple paths around a single mass become possible, e.g. one around the left and one around the right side of the deflector. The observer, who will see an image of the source along the backward tangent of each ray arriving at his position, will then see multiple images of a single source;

- in addition, the light deflection of two neighbouring rays may be different. Suppose a pair of rays, one from one side and one from the other side of a source, passes by a lensing mass distribution. The ray which passes closer to the deflector will be bent more than the other, thus the source will appear stretched. It is thus expected that gravitational lensing will typically distort the sources. By the same mechanism, they can appear larger or smaller than they originally are;

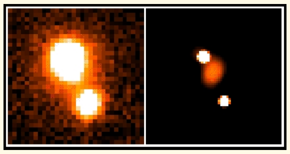
- since photons are not created, neither destroyed by the lensing effect the surface brightness of the source will remain unchanged. Since, as we said, the size is not conserved, this implys that the source can be either *magnified* or *demagnified* by lensing. If it is enlarged it will appear brigher, otherwise fainter;

- in case that multiple light paths are possible between the source and the observer, since they will be characterized by different lengths, the light travel times will differ for the different images. One of the images will appear first, the others will be delayed.

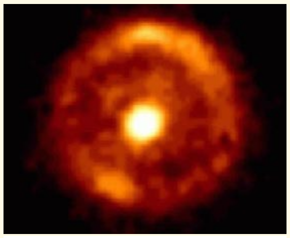
Starting from the equivalence principle, we are thus arrived at the expectation of multiple images, distortions, magnification, and time delayes. All of these phenomena have been observed in numerous cases.

Example: Multiply-imaged quasars

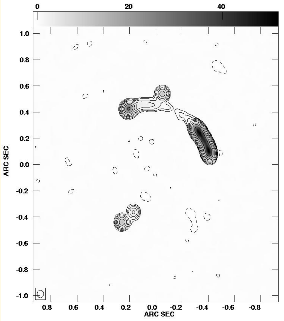
Identification of the lensing galaxy in a double quasar system: the left panel shows on infrared (J-band) observation of the two images of double quasar HE 1104-1825 ($z_Q = 2.316$, $\theta = 3.2''$). The right panel obtained with some new deconvolution technique nicely reveals the lensing galaxy (at $z_G = 1.66$) between the quasar images (Credits: European Southern Observatory).



Example: Einstein ring



B1938+666 is another multiple-image lens, and was discovered in JVAS (Jodrell/VLA Astrometric Survey). This is a survey of flat-spectrum radio sources designed to identify gravitational lens candidates. HST observations show an Einstein ring in IR. The lens redshift is 0.878, but the source redshift is not yet known (IR spectroscopy required).

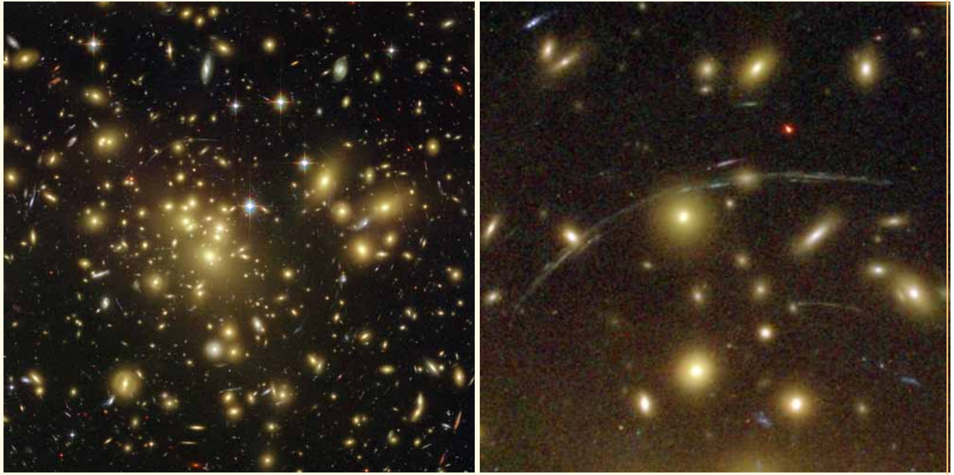


The bottom figure shows a MERLIN image of this system at 5GHz. In radio there is a significant arc visible.

Credit: JVAS/CLASS

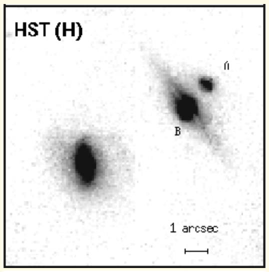
Example: Arcs in galaxy clusters

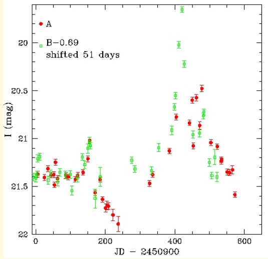
Abell 1689 is a galaxy cluster at z=0.183. The gravity of the cluster's trillion stars - plus dark matter - acts as a 2-million-light-year-wide 'lens' in space. This 'gravitational lens' bends and magnifies the light of galaxies located far behind it, distorting their shapes and creating multiple images of individual galaxies.



Credit: NASA, N. Benitez (JHU), T. Broadhurst (The Hebrew University), H. Ford (JHU), M. Clampin(STScI), G. Hartig (STScI), G. Illingworth (UCO/Lick Observatory), the ACS Science Team and ESA.

Example: Time delays





B1600+434 is a double gravitational lens system. A distant QSO at redshift $z = 1.59$ is lensed by an edge-on-late-type galaxy at $z = 0.41$ and has two images, labeled with A and B in the upper image. QSO's are characterized by intrinsic variability of their luminosity. The light curves of the two images have the same shape, as expected since they arise from the same source. However, the light curve of the image B is shifted by ~ 50 days with respect to that of image A. The reason is the different path of the light coming from the two images.

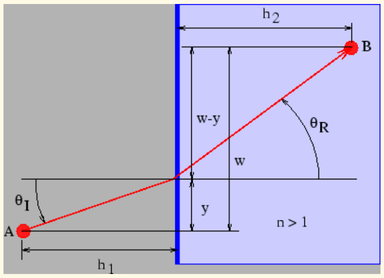
Credit: I. Burud, Institut d'Astrophysique et de Gophysique de Lige, Avenue de Cointe 5, B-4000 Lige, Belgium

1.2 Fermat's principle and light deflection

Starting from the field equations of general relativity, light deflection can be calculated by studying geodesic curves. It turns out that light deflection can equivalently be described by Fermat's principle, as in geometrical optics. This will be our starting point.

Example: Fermat's Principle in geometrical optics

In its simplest form the Fermat's principle says that light waves of a given frequency traverse the path between two points which takes the least time. The speed of light in a medium with refractive index n is c/n, where c is its speed in a vacuum. Thus, the time required for light to go some distance in such a medium is n times the time light takes to go the same distance in a vacuum.



Referring to the figure above, the time required for light to go from A to B becomes

$$t = [\{h_1^2 + y^2\}^{1/2} + n\{h_2^2 + (w-y)^2\}^{1/2}]/c.$$

We find the minimum time by differentiating t with respect to y and setting the result to zero, with the result that

$$\frac{y}{\{h_1^2 + y^2\}^{1/2}} = n\frac{w-y}{\{h_2^2 + (w-y)^2\}^{1/2}}.$$

However, we note that the left side of this equation is simply $\sin\theta_I$, while the right side is $n\sin\theta_R$, so that the minimum time condition reduces to

$$\sin\theta_I = n\sin\theta_R$$

We recognize this result as Snell's law.

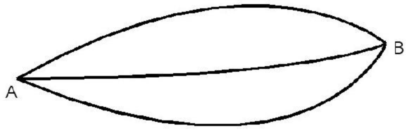
We first need an index of refraction n because Fermat's principle says that light will follow a path along which the travel time,

$$\int \frac{n}{c} \mathrm{d}l \ , \tag{1.14}$$

will be extremal. As in geometrical optics, we thus search for a path, $\vec{x}(l)$, for which the variation

$$\delta \int_A^B n(\vec{x}(l)) \mathrm{d}l = 0 \ , \tag{1.15}$$

where the starting point A and the end point B are kept fixed.



In order to find the index of refraction, we make a first approximation: we assume that the lens is weak, and that it is small compared to the overall dimensions of the optical system composed of source, lens and observer. With "weak lens", we mean a lens whose Newtonian gravitational potential Φ is much smaller than c^2, $\Phi/c^2 \ll 1$. Note that this approximation is valid in virtually all cases of astrophysical interest. Consider for istance a galaxy cluster: its gravitational potential is $|\Phi| < 10^{-4}c^2 \ll c^2$.

The metric of unperturbed space-time is the Minkowski metric,

$$\eta_{\mu\nu} = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 \\ 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & -1 \end{pmatrix} \ ,$$

whose line element is

$$\mathrm{d}s^2 = \eta_{\mu\nu} \mathrm{d}x^\mu \mathrm{d}x^\nu = (\mathrm{d}x^0)^2 - (\mathrm{d}\vec{x})^2 = c^2 \mathrm{d}t^2 - (\mathrm{d}\vec{x})^2 \ . \tag{1.16}$$

A weak lens perturbs this metric such that

$$\eta_{\mu\nu} \to g_{\mu\nu} = \begin{pmatrix} 1+\frac{2\Phi}{c^2} & 0 & 0 & 0 \\ 0 & -(1-\frac{2\Phi}{c^2}) & 0 & 0 \\ 0 & 0 & -(1-\frac{2\Phi}{c^2}) & \\ 0 & 0 & 0 & -(1-\frac{2\Phi}{c^2}) \end{pmatrix}$$

for which the line element becomes

$$\mathrm{d}s^2 = g_{\mu\nu} \mathrm{d}x^\mu \mathrm{d}x^\nu = \left(1+\frac{2\Phi}{c^2}\right) c^2 \mathrm{d}t^2 - \left(1-\frac{2\Phi}{c^2}\right) (\mathrm{d}\vec{x})^2 \ . \tag{1.17}$$

Now light propagates at zero eigentime, $\mathrm{d}s = 0$, from which we gain

$$\left(1+\frac{2\Phi}{c^2}\right) c^2 \mathrm{d}t^2 = \left(1-\frac{2\Phi}{c^2}\right) (\mathrm{d}\vec{x})^2 \ . \tag{1.18}$$

The light speed in the gravitational field is thus

$$c' = \frac{|\mathrm{d}\vec{x}|}{\mathrm{d}t} = c\sqrt{\frac{1+\frac{2\Phi}{c^2}}{1-\frac{2\Phi}{c^2}}} \approx c\left(1+\frac{2\Phi}{c^2}\right) \ , \tag{1.19}$$

where we have used that $\Phi/c^2 \ll 1$ by assumption. The index of refraction is thus

$$n = c/c' = \frac{1}{1+\frac{2\Phi}{c^2}} \approx 1 - \frac{2\Phi}{c^2} \; . \tag{1.20}$$

With $\Phi \leq 0$, $n \geq 1$, and the light speed c' is lower than in vacuum.

n will typically depend on the spatial coordinate $\vec{x}$ and perhaps also on time t. Let $\vec{x}(l)$ be a light path. Then the light travel time is proportional to

$$\int_A^B n[\vec{x}(l)]\mathrm{d}l \; , \tag{1.21}$$

and the light path follows from

$$\delta \int_A^B n[\vec{x}(l)]\mathrm{d}l = 0 \; . \tag{1.22}$$

This is a standard variational problem, which leads to the well known Euler equations. In our case we write

$$\mathrm{d}l = \left|\frac{\mathrm{d}\vec{x}}{\mathrm{d}\lambda}\right| \mathrm{d}\lambda \; , \tag{1.23}$$

with a curve parameter λ which is yet arbitrary, and find

$$\delta \int_{\lambda_A}^{\lambda_B} \mathrm{d}\lambda \, n[\vec{x}(\lambda)] \left|\frac{\mathrm{d}\vec{x}}{\mathrm{d}\lambda}\right| = 0 \tag{1.24}$$

The expression

$$n[\vec{x}(\lambda)] \left|\frac{\mathrm{d}\vec{x}}{\mathrm{d}\lambda}\right| \equiv L(\dot{\vec{x}}, \vec{x}, \lambda) \tag{1.25}$$

takes the role of the Lagrangian in analytic mechanics, with

$$\dot{\vec{x}} \equiv \frac{\mathrm{d}\vec{x}}{\mathrm{d}\lambda} \; . \tag{1.26}$$

Finally, we have

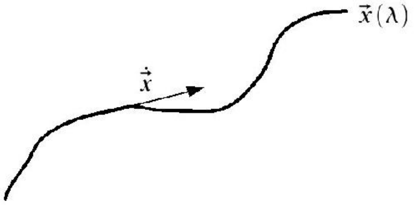
$$\left|\frac{\mathrm{d}\vec{x}}{\mathrm{d}\lambda}\right| = |\dot{\vec{x}}| = (\dot{\vec{x}}^2)^{1/2} \; . \tag{1.27}$$

Using these expressions, we find the Euler equations

$$\frac{\mathrm{d}}{\mathrm{d}\lambda}\frac{\partial L}{\partial \dot{\vec{x}}} - \frac{\partial L}{\partial \vec{x}} = 0 \; . \tag{1.28}$$

Now,

$$\frac{\partial L}{\partial \vec{x}} = |\dot{\vec{x}}|\frac{\partial n}{\partial \vec{x}} = (\vec{\nabla} n)|\dot{\vec{x}}| \; , \; \frac{\partial L}{\partial \dot{\vec{x}}} = n\frac{\dot{\vec{x}}}{|\dot{\vec{x}}|} \; . \tag{1.29}$$



Evidently, $\dot{\vec{x}}$ is a tangent vector to the light path , which we can assume to be normalized by a suitable choice for the curve parameter λ. We thus assume $|\dot{\vec{x}}| = 1$ and write $\vec{e} \equiv \dot{\vec{x}}$ for the unit tangent vector to the light path. Then, we have

$$\frac{\mathrm{d}}{\mathrm{d}\lambda}(n\vec{e}) - \vec{\nabla} n = 0 \ , \tag{1.30}$$

or

$$n\dot{\vec{e}} + \vec{e} \cdot [(\vec{\nabla} n)\dot{\vec{x}}] = \vec{\nabla} n \ , \tag{1.31}$$

$$\Rightarrow n\dot{\vec{e}} = \vec{\nabla} n - \vec{e}(\vec{\nabla} n \cdot \vec{e}) \ . \tag{1.32}$$

The second term on the right hand side is the derivative along the light path, thus the whole right hand side is the gradient of n perpendicular to the light path. Thus

$$\dot{\vec{e}} = \frac{1}{n}\vec{\nabla}_\perp n = \vec{\nabla}_\perp \ln n \ . \tag{1.33}$$

As $n = 1 - 2\Phi/c^2$ and $\Phi/c^2 \ll 1$, $\ln n \approx -2\Phi/c^2$, and

$$\dot{\vec{e}} \approx -\frac{2}{c^2}\vec{\nabla}_\perp \Phi \ . \tag{1.34}$$

The total deflection angle of the light path is now the integral over $-\dot{\vec{e}}$ along the light path,

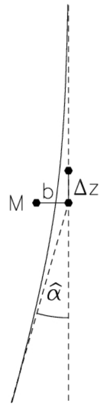
$$\hat{\vec{\alpha}} = \frac{2}{c^2}\int_{\lambda_A}^{\lambda_B} \vec{\nabla}_\perp \Phi \mathrm{d}\lambda \ . \tag{1.35}$$

The deflection is thus the integral over the "pull" of the gravitational potential perpendicular to the light path. Note that $\vec{\nabla}\Phi$ points away from the lens centre, so $\hat{\vec{\alpha}}$ points towards it.

As it stands, the equation for $\hat{\vec{\alpha}}$ is not useful, as we would have to integrate over the actual light path. However, since $\Phi/c^2 \ll 1$, we expect the deflection angle to be small. Then, we can adopt the Born approximation familiar from scattering theory and integrate over the unperturbed light path.

Suppose, therefore, a light ray starts out into $+\vec{e}_z$-direction and passes a lens at $z = 0$, with impact parameter b. The deflection angle is then given by

$$\hat{\vec{\alpha}}(b) = \frac{2}{c^2}\int_{-\infty}^{+\infty} \vec{\nabla}_\perp \phi \mathrm{d}z \tag{1.36}$$



Special case: point mass lens

If the lens is a point mass, then

$$\Phi = -\frac{GM}{r} \tag{1.37}$$

with $r = \sqrt{x^2 + y^2 + z^2} = \sqrt{b^2 + z^2}$, $b = \sqrt{x^2 + y^2}$ and

$$\vec{\nabla}_\perp \phi = \begin{pmatrix} \partial_x \Phi \\ \partial_y \Phi \end{pmatrix} = \frac{GM}{r^3} \begin{pmatrix} x \\ y \end{pmatrix} . \tag{1.38}$$

The deflection angle is then

$$\begin{aligned} \hat{\vec{\alpha}}(b) &= \frac{2GM}{c^2} \begin{pmatrix} x \\ y \end{pmatrix} \int_{-\infty}^{+\infty} \frac{\mathrm{d}z}{(b^2 + z^2)^{3/2}} \\ &= \frac{4GM}{c^2} \begin{pmatrix} x \\ y \end{pmatrix} \left[\frac{z}{b^2 (b^2 + z^2)^{1/2}} \right]_0^\infty = \frac{4GM}{c^2 b} \begin{pmatrix} \cos\phi \\ \sin\phi \end{pmatrix} , \end{aligned} \tag{1.39}$$

with

$$\begin{pmatrix} x \\ y \end{pmatrix} = b \begin{pmatrix} \cos\phi \\ \sin\phi \end{pmatrix} \tag{1.40}$$

Notice that $R_s = \frac{2GM}{c^2}$ is the Schwarzschild radius of a (point) mass M, thus

$$|\hat{\vec{\alpha}}| = \frac{4GM}{c^2 b} = 2 \frac{R_s}{b} . \tag{1.41}$$

Also notice that $\hat{\vec{\alpha}}$ is linear in M, thus the deflection angles of an array of lenses can linearly be superposed.
Note that the deflection angle found here in the framework of general relativity exceeds by a factor of two that calculated by using standard Newtonian Gravity (see Eq. 1.13), as anticipated at the beginning of this chapter.

Since the speed of the light is reduced in the gravitational field, $c' = c/n$, the travel time (along the perturbed path) is larger by

$$\Delta t = \int \frac{\mathrm{d}l}{c'} - \int \frac{\mathrm{d}l}{c} = \int (n-1) \mathrm{d}l = -\frac{2}{c^3} \int \Phi \mathrm{d}l . \tag{1.42}$$

This is the so-called *Shapiro delay* (Shapiro, 1964).

Bibliography

Einstein, A., 1916, *Ann. der Ph.*, **49**, 769

Shapiro, I. I., 1964, *Phys. Rev. Lett.*, **13**, 789

Soldner, J., 1801, *Berliner Astronomisches Jahrbruch fuer Jahr 1804*

Will, C. M., 1988, *Am. J. Phys.*, **56**, 5