Cosmological constant and friends

P. Binétruy
binetruy@apc.univ-paris7.fr
AstroParticule et Cosmologie[1], *Université Paris Diderot,*
Bâtiment Condorcet, 10, rue Alice Domon et Léonie Duquet,
F-75205 Paris Cedex 13, France

Abstract

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Contents

Chapter 1

Introducing the cosmological constant

1.1 The first days

The evolution of the universe at large is governed by gravity, and thus described by Einstein's equations. We recall that, in the context of general relativity, the metric is a dynamical field, i.e. is spacetime dependent: its fluctuating part is the gravitational field. Einstein's equations are highly non-linear second order differential equations of motion for this field. They read[1]:

$$G_{\mu\nu} \equiv R_{\mu\nu} - \frac{1}{2} g_{\mu\nu} R = 8\pi G_{\scriptscriptstyle N} T_{\mu\nu} + \lambda g_{\mu\nu} \ . \tag{1.1}$$

[1] The metric signature we adopt throughout is Einstein's choice: $(+,-,-,-)$.

where $R_{\mu\nu}$ is the Ricci tensor, R the associated curvature scalar[2] and $T_{\mu\nu}$ the energy-momentum tensor; finally λ is the cosmological constant which has the dimension of an inverse length squared.

Thus Einstein's equations relate the geometry of spacetime (the left-hand side of (1.1)) with its matter field content (the right-hand side). As we will see later, it is still an open question whether the cosmological term belongs to the left or the right-hand side.

One may try to apply Einstein's equations to describe not a given gravitational system like a planet or a star but the evolution of the whole universe. In Einstein's days, this was a bold move: it should be remembered how little of the universe was known at the time these equations were written. "In 1917, the world was supposed to consist of our galaxy and presumably a void beyond. The Andromeda nebula had not yet been certified to lie beyond the Milky Way."[Pais [23] p. 286] Indeed, it is in this context that Einstein introduced the cosmological constant in order to have a static solution (until it was observed by Hubble that the universe is expanding) for the universe.

More precisely [23], Einstein first noticed that a slight modification of the Poisson equation $\Delta\Phi = 4\pi G_N \rho$, namely

$$\Delta\Phi - \lambda\Phi = 4\pi G_N \rho \tag{1.5}$$

allowed a solution with a constant density ρ ($\Phi = -4\pi G_N \rho/\lambda$) and thus a static Newtonian universe. In the context of general relativity, he found a

[2] In the context of general relativity, one defines the Christoffel symbol or spin connection $\Gamma^\rho{}_{\mu\nu}$ which is the analogue of the gauge field (it appears in covariant derivatives). It is defined in terms of the metric as:

$$\Gamma^\rho{}_{\mu\nu} = \frac{1}{2} g^{\rho\sigma} \left[\partial_\mu g_{\nu\sigma} + \partial_\nu g_{\mu\sigma} - \partial_\sigma g_{\mu\nu}\right] \quad , \tag{1.2}$$

where $g^{\rho\sigma}$ is the inverse metric tensor: $g^{\rho\sigma} g_{\sigma\tau} = \delta^\rho_\tau$.

In the same way that one defines the field strength by differentiating the gauge field, one introduces the Riemann curvature tensor:

$$R^\mu{}_{\nu\alpha\beta} = \partial_\alpha \Gamma^\mu{}_{\nu\beta} - \partial_\beta \Gamma^\mu{}_{\nu\alpha} + \Gamma^\mu{}_{\alpha\sigma}\Gamma^\sigma{}_{\nu\beta} - \Gamma^\mu{}_{\beta\sigma}\Gamma^\sigma{}_{\nu\alpha} \quad . \tag{1.3}$$

By contracting indices, one then defines the Ricci tensor $R_{\mu\nu}$ and the curvature scalar R

$$R_{\mu\nu} \equiv R^\alpha{}_{\mu\alpha\nu}, \quad , \quad R \equiv g^{\mu\nu} R_{\mu\nu} \quad . \tag{1.4}$$

static solution of (1.1) under the condition that

$$\lambda = \frac{1}{r^2} = 4\pi G_{_N} \rho \ , \tag{1.6}$$

where r is the spatial curvature (see Exercise 1-1). It was soon shown that this Einstein universe is unstable to small perturbations. The blow was the discovery by Hubble [20] that the Universe is expanding. This could be described by the expanding solutions to Einstein's equations found by Friedmann in 1922 [17].

<u>Exercise 1-1</u> : Consider the following metric

$$g_{00} = 1 \ , \quad g_{ij} = -\delta_{ij} + \frac{x_i x_j}{x^2 - r^2} \ , \ x^2 = \sum_{i=1}^{3} x_i^2 \ . \tag{1.7}$$

a) Show that it is a solution of Einstein's equations (1.1) in the case of non-relativistic matter with a constant energy density ρ satisfying the condition (1.6).

b) Prove that, in the Newtonian limit, one recovers (1.5).

Hints: a) ${\Gamma^i}_{jk} = r^{-2} \left[x_i \delta_{jk} - x_i x_j x_k / (x^2 - r^2) \right]$ which gives $R_{ij} = -2 g_{ij} / r^2$.

b) In the Newtonian limit, $G_{00} \sim \Delta g_{00}$ with $g_{00} = 1 + 2\Phi$.

Under the assumption that the Universe is homogeneous and isotropic on scales of order 100 Mpc (1 pc = 3.262 light-year = 3.086×10^{16} m) and larger, one may first try to find a homogeneous and isotropic metric as a solution of Einstein's equations. The most general ansatz is, up to coordinate redefinitions, the Robertson-Walker metric:

$$\begin{aligned} ds^2 &= c^2 dt^2 - a^2(t) \ \gamma_{ij} dx^i dx^j, & (1.8) \\ \gamma_{ij} dx^i dx^j &= \frac{dr^2}{1 - kr^2} + r^2 \left(d\theta^2 + \sin^2\theta d\phi^2 \right), & (1.9) \end{aligned}$$

where $a(t)$ is the cosmic scale factor, which is time-dependent in an expanding or contracting universe. Such a universe is called a Friedmann-Lemaître universe. The constant k which appears in the spatial metric γ_{ij} can take the values ± 1 or 0: the value 0 corresponds to flat space, i.e. usual Minkowski spacetime; the value $+1$ to closed space ($r^2 < 1$) and the value -1 to open space. Note that r is dimensionless whereas a has the dimension of a length. From now on, we set $c = 1$, except otherwise stated.

The components of the Einstein tensor now read:

$$\begin{aligned} G_{tt} &= 3\left(\frac{\dot{a}^2}{a^2}+\frac{k}{a^2}\right), && (1.10) \\ G_{ij} &= -\gamma_{ij}\left(\dot{a}^2+2a\ddot{a}+k\right), && (1.11) \end{aligned}$$

where we use standard notations: $\dot{a}$ is the first time derivative of the cosmic scale factor, $\ddot{a}$ the second time derivative.

Exercise 1-2 : In the case of the Robertson-Walker metric (1.8),
a) compute the non-vanishing Christoffel symbols (1.2),
b) using the fact that the Ricci tensor associated with the 3-dimensional metric γ_{ij} is simply $R_{ij}(\gamma) = 2k\gamma_{ij}$, compute the components of the Ricci tensor and the scalar curvature (1.4),
c) deduce the components of the Einstein tensor (1.10) and (1.11).

Hints: a) $\Gamma^i{}_{jt} = \delta^i_j \dot{a}/a$, $\Gamma^t{}_{ij} = a\dot{a}\gamma_{ij}$, $\Gamma^i{}_{jk} = \Gamma^i{}_{jk}(\gamma)$.
b) $R_{tt} = -3\ddot{a}/a$, $R_{ij} = (2k+\ddot{a}a+2\dot{a}^2)\,\gamma_{ij}$, $R = -6\,(k+\ddot{a}a+\dot{a}^2)\,/a^2$.

For the energy-momentum tensor, we follow our assumption of homogeneity and isotropy and assimilate the content of the Universe to a perfect fluid:

$$T_{\mu\nu} = -pg_{\mu\nu} + (p+\rho)U_\mu U_\nu \quad , \tag{1.12}$$

where U^μ is the velocity 4-vector ($U^t = 1, U^i = 0$). It follows from (2.4) that $T_{tt} = \rho$ and $T_{ij} = a^2 p\gamma_{ij}$. The pressure p and energy density ρ usually satisfy the equation of state:

$$p = w\rho \quad . \tag{1.13}$$

The constant w takes the value $w \sim 0$ for non-relativistic matter (negligible pressure) and $w = 1/3$ for relativistic matter (radiation). In all generality, the perfect fluid consists of several components with different values of w.

One now obtains from the $(0,0)$ and (i,j) components of the Einstein equations (1.1):

$$\begin{aligned} 3\left(\frac{\dot{a}^2}{a^2}+\frac{k}{a^2}\right) &= 8\pi G_{\scriptscriptstyle N}\rho+\lambda, && (1.14) \\ \dot{a}^2+2a\ddot{a}+k &= -8\pi G_{\scriptscriptstyle N} a^2 p + a^2\lambda, && (1.15) \end{aligned}$$

The first of the preceding equations can be written as the Friedmann equation, which gives an expression for the Hubble parameter $H \equiv \dot{a}/a$ measuring the rate of the expansion of the Universe:

$$H^2 \equiv \frac{\dot{a}^2}{a^2} = \frac{1}{3}\left(\lambda + 8\pi G_{\scriptscriptstyle N}\rho\right) - \frac{k}{a^2} \ . \tag{1.16}$$

Note that the cosmological constant appears as a constant contribution to the Hubble parameter.

Friedmann equation should be supplemented by the conservation of the energy-momentum tensor which simply yields:

$$\dot{\rho} = -3H(p+\rho) \quad . \tag{1.17}$$

Hence a component with equation of state (1.13) has its energy density scaling as $\rho \sim a(t)^{-3(1+w)}$. Thus non-relativistic matter (often referred to as matter) energy density scales as a^{-3}. In other words, the energy density of matter evolves in such a way that ρa^3 remains constant. Radiation scales as a^{-4} and a component with equation of state $p = -\rho$ ($w = -1$) has constant energy density[3].

We note for future use that, if a component with equation of state (1.13) dominates the energy density of the universe (as well as the curvature term $-k/a^2$), then (1.16) has a scaling solution

$$a(t) \sim t^{\nu} \ , \quad \text{with } \nu = \frac{2}{3(1+w)} \ . \tag{1.18}$$

For example, in a matter-dominated universe, $a(t) \sim t^{2/3}$.

1.2 The Hubble constant and the cosmological constant

The Friedmann equation allows to define the Hubble constant H_0, i.e. the present value of the Hubble parameter, which sets the scale of our Universe at present time. Because of the troubled history of the measurement

[3] The latter case corresponds to a cosmological constant as can be seen from (1.14-1.15) where the cosmological constant can be replaced by a component with $\rho_\Lambda = -p_\Lambda = \lambda/(8\pi G_{\scriptscriptstyle N})$.

of the Hubble constant, it has become customary to express it in units of 100 km.s^{-1}.Mpc^{-1} which gives its order of magnitude. Present measurements give

$$h_0 \equiv \frac{H_0}{100 \text{ km.s}^{-1}.\text{Mpc}^{-1}} = 0.7 \pm 0.1 \quad .$$

The corresponding length and time scales are:

$$\ell_{H_0} \equiv \frac{c}{H_0} = 3000 \ h_0^{-1} \text{ Mpc} = 9.25 \times 10^{25} \ h_0^{-1} \text{ m}, \tag{1.19}$$

$$t_{H_0} \equiv \frac{1}{H_0} = 3.1 \times 10^{17} \ h_0^{-1} \text{ s} = 9.8 \ h_0^{-1} \text{ Gyr}. \tag{1.20}$$

A reference energy density at present time t_0 is obtained from the Friedmann equation for vanishing cosmological ($\lambda = 0$) and flat space ($k = 0$):

$$\rho_c \equiv \frac{3H_0^2}{8\pi G_{_N}} = 1.9 \ 10^{-26} \ h_0^2 \text{ kg/m}^3 \quad . \tag{1.21}$$

This corresponds to approximately one galaxy per Mpc^3 or 5 protons per m^3. In fundamental units where $\hbar = c = 1$, this is of the order of $(10^{-3}\text{eV})^4$. In the case of a vanishing cosmological constant, it follows from (4.1) that, depending on whether the present energy density of the Universe ρ_0 is larger, equal or smaller than ρ_c, the present Universe is spatially open ($k > 0$), flat ($k = 0$) or closed ($k < 0$). Hence the name critical density for ρ_c.

It has become customary to normalize the different forms of energy density in the present Universe in terms of this critical density. Separating the energy density $\rho_{_{M0}}$ presently stored in non-relativistic matter (baryons, neutrinos, dark matter,...) from the density $\rho_{_{R0}}$ presently stored in radiation (photons, relativistic neutrino if any), one defines:

$$\Omega_{_M} \equiv \frac{\rho_{_{M0}}}{\rho_c}, \quad \Omega_{_R} \equiv \frac{\rho_{_{R0}}}{\rho_c}, \quad \Omega_\Lambda \equiv \frac{\lambda}{3H_0^2}, \quad \Omega_k \equiv -\frac{k}{a_0^2 H_0^2} \ . \tag{1.22}$$

The last term comes from the spatial curvature and is not strictly speaking a contribution to the energy density.

Then the Friedmann equation taken at time t_0 simply reads

$$\Omega_{_M} + \Omega_{_R} + \Omega_\Lambda = 1 - \Omega_k \ . \tag{1.23}$$

Since matter dominates over radiation in the present Universe, we may neglect $\Omega_{_R}$ in the preceding equation. Using the dependence of the different components with the scale factor $a(t) = a_0/(1+z)$, one may then rewrite the Friedmann equation at any time as:

$$H^2(t) = H_0^2 \left[\Omega_\Lambda + \Omega_{_M}\left(\frac{a_0}{a(t)}\right)^3 + \Omega_{_R}\left(\frac{a_0}{a(t)}\right)^4 + \Omega_k\left(\frac{a_0}{a(t)}\right)^2\right] \quad (1.24)$$

$$\text{or } H^2(z) = H_0^2 \left[\Omega_{_M}(1+z)^3 + \Omega_{_R}(1+z)^4 + \Omega_k(1+z)^2 + \Omega_\Lambda\right] . \quad (1.25)$$

where a_0 is the present value of the cosmic scale factor and all time dependences (or alternatively redshift dependence) have been written explicitly. We note that, even if $\Omega_{_R}$ is negligible in (1.23), this is not so in the early Universe because the radiation term increases faster than the matter term in (1.24) as one gets back in time (i.e. as $a(t)$ decreases). If we add an extra component X with equation of state $p_{_X} = w_{_X}\rho_{_X}$, it contributes an extra term $\Omega_{_X}\,(a_0/a(t))^{3(1+w_X)}$ where $\Omega_{_X} = \rho_{_X}/\rho_c$.

Differentiating the Friedmann equation with respect to time, and using the energy-momentum conservation (1.17), one easily obtains

$$\ddot{a} = -\frac{4\pi G_{_N}}{3} a(3p+\rho) + a\frac{\lambda}{3} . \quad (1.26)$$

This allows to recover (1.15) from Friedmann equation and energy-momentum conservation.

The acceleration of our universe is usually measured by the deceleration parameter q which is defined as:

$$q \equiv -\frac{\ddot{a}a}{\dot{a}^2} . \quad (1.27)$$

Using (1.26) and separating again matter and radiation, we may write it at present time t_0 as:

$$q_0 = -\frac{1}{H_0^2}\left(\frac{\ddot{a}}{a}\right)_{t=t_0} = \frac{1}{2}\Omega_{_M} + \Omega_{_R} - \Omega_\Lambda . \quad (1.28)$$

Once again, the radiation term $\Omega_{_R}$ can be neglected in this relation. We see that in order to have an acceleration of the expansion ($q_0 < 0$), we need the cosmological constant to dominate over the other terms.

We can also write the deceleration parameter in (1.27) in terms of redshift as in (1.25)

$$q = \frac{H_0^2}{2H(z)^2}\left[\Omega_{_M}(1+z)^3 + 2\Omega_{_R}(1+z)^4 - 2\Omega_\Lambda\right] . \quad (1.29)$$

This shows that the universe starts accelerating at redshift values $1 + z \sim (2\Omega_\Lambda/\Omega_{_M})^{1/3}$ (neglecting $\Omega_{_R}$), that is typically redshifts of order 1.

The measurements of the Hubble constant and of the deceleration parameter at present time allow to obtain the behaviour of the cosmic scale factor in the last stages of the evolution of the universe:

$$a(t) = a_0 \left[1 + \frac{t - t_0}{t_{H_0}} - \frac{q_0}{2} \frac{(t - t_0)^2}{t_{H_0}^2} + \cdots \right] . \tag{1.30}$$

Typically, since we know that the spatial curvature term certainly does not represent presently a dominant contribution to the expansion of the Universe, (1.16) considered at present time implies the following constraint on λ (barring a cancellation between the dynamical ρ and the constant λ):

$$|\lambda| \leq H_0^2 . \tag{1.31}$$

In other words, the length scale $\ell_\Lambda \equiv |\lambda|^{-1/2}$ associated with the cosmological constant must be larger than the Hubble length $\ell_{H_0} \equiv cH_0^{-1} = h_0^{-1}.10^{26}$ m, and thus be a cosmological distance.

This is not a problem as long as one remains classical: ℓ_{H_0} provides a natural cosmological scale for our present Universe. The problem arises when one tries to combine gravity with quantum theory. Indeed, from Newton's constant *and* the Planck constant $\hbar$, we have seen that we can construct the Planck mass scale $m_{_P} = \sqrt{\hbar c/(8\pi G_{_N})} = 2.4 \times 10^{18}$ GeV/c^2. The corresponding length scale is the Planck length

$$\ell_{_P} = \frac{\hbar}{m_{_P} c} = 8.1 \times 10^{-35} \text{ m} . \tag{1.32}$$

The above constraint now reads:

$$\ell_\Lambda \equiv |\lambda|^{-1/2} \geq \ell_{H_0} = \frac{c}{H_0} \sim 10^{60} \ \ell_{_P}. \tag{1.33}$$

In other words, there are more than sixty orders of magnitude between the scale associated with the cosmological constant and the scale of quantum gravity.

1.3 Cosmological constant and vacuum energy

A rather obvious solution is to take $\lambda = 0$. This is as valid a choice as any other in a pure gravity theory. Unfortunately, it is an unnatural one when one introduces any kind of matter. Indeed, set λ to zero but assume that there is a non-vanishing vacuum (*i.e.* ground state) energy: $< T_{\mu\nu} >= \rho_{\rm vac} g_{\mu\nu}$; then the Einstein equations (1.1) read

$$R_{\mu\nu} - \frac{1}{2} g_{\mu\nu} R = 8\pi G_{_N} T_{\mu\nu} + 8\pi G_{_N} \rho_{\rm vac} g_{\mu\nu} \ . \tag{1.34}$$

The last term is interpreted as an effective cosmological constant (from now on, we set $\hbar = c = 1$) [33]:

$$\lambda_{\rm eff} = 8\pi G_{_N} \rho_{\rm vac} \equiv \frac{\Lambda^4}{m_{_P}^2} \ . \tag{1.35}$$

Generically, $\rho_{\rm vac}$ receives a non-zero contribution from symmetry breaking: for instance, the scale Λ would be typically of the order of 100 GeV in the case of the electroweak gauge symmetry breaking or 1 TeV in the case of supersymmetry breaking. But the constraint (1.33) now reads:

$$\Lambda \leq 10^{-30}\ m_{_P} \sim 10^{-3}\ {\rm eV}. \tag{1.36}$$

It is this very unnatural fine-tuning of parameters (in explicit cases $\rho_{\rm vac}$ and thus Λ are functions of the parameters of the theory) that is referred to as the cosmological constant problem, or more accurately the *vacuum energy problem.*

1.4 Supersymmetry

The most natural reason why vacuum energy would be vanishing is a symmetry argument. Global supersymmetry indeed provides such a rationale. Let us recall briefly how this arises.

The supersymmetry algebra expresses the fact that the commutator of two supersymmetry transformations (generator Q_r, r spinor index) is a space-time translation (generator P_μ). More precisely,

$$\{Q_r, \bar{Q}_s\} = 2\gamma^\mu_{rs}\ P_\mu \ , \tag{1.37}$$

where $\bar{Q} \equiv Q\gamma^0$ and the anticommutator arises from the fact that the supersymmetry transformation parameter is an anticommuting spinor. Since the generator of time translations P_0 is the Hamiltonian H, we may easily infer from (1.37) an expression for the Hamiltonian of the system. Indeed, it reads explicitly

$$\{Q_r, Q_t\}\gamma^0_{ts} = 2\gamma^\mu_{rs}\ P_\mu \ . \tag{1.38}$$

Contracting with γ^0_{sr}, one obtains

$$\sum_{r,t} \{Q_r, Q_t\} \left[(\gamma^0)^2\right]_{tr} = 2\ \mathrm{Tr}\left(\gamma^0\gamma^\mu\right) P_\mu \quad . \tag{1.39}$$

Using $(\gamma^0)2 = \mathbf{1}$ and $\mathrm{Tr}(\gamma^0\gamma^\mu) = 4g^{0\mu}$, one obtains

$$\sum_r Q_r^2 = 4P^0 = 4P_0 \quad . \tag{1.40}$$

Thus

$$H = \frac{1}{4}\sum_r Q_r^2 \quad . \tag{1.41}$$

It is easy to infer the following expression for the vacuum energy:

$$\langle 0|H|0\rangle = \frac{1}{4}\sum_r \parallel Q_r|0\rangle \parallel^2 \ . \tag{1.42}$$

Thus, the vacuum energy vanishes if and only if supersymmetry is a symmetry of the vacuum: $Q_r|0\rangle = 0$ for all r.[4]

The problem however is that, at the same time, supersymmetry predicts equal boson and fermion masses and therefore needs to be broken. The amount of breaking necessary to push the supersymmetric partners high enough not to have been observed yet, is incompatible with the limit (1.36).

Moreover, in the context of cosmology, we should consider supersymmetry in a gravity context and thus work with its local version, supergravity (following (1.37), local supersymmetry transformations are associated with local spacetime translations which are nothing but the reparametrizations which play a central role in general relativity). In this context, the criterion of vanishing vacum energy is traded for one of vanishing mass for the gravitino, the supersymmetric partner of the graviton[5]. Local supersymmetry is

[4]Remember that a supersymmetry transformation U is obtained by exponentiating the generators: $U|0\rangle = |0\rangle$.

[5]More precisely, one can write the following term invariant under supersymmetry:

$$\tilde{S} = \int d^4x\sqrt{-g}\left[3m^2_{3/2}m^2_P - m_{3/2}\bar{\psi}_\mu\sigma^{\mu\nu}\psi_\nu\right] \tag{1.43}$$

then absolutely compatible with a non-vanishing vacuum energy, preferably a negative one (although possibly also a positive one). This is both a blessing and a problem: supersymmetry may be broken while the cosmological constant remains small, but we have lost our rationale for a vanishing, or very small, cosmological constant and fine-tuning raises again its ugly head.

In some supergravity theories however, one may recover the vanishing vacuum energy criterion.

1.5 Observations

A very diverse set of cosmological data converges towards the observation that the Universe has been undergoing in the recent past (for redshifts of order one or smaller) an acceleration of its expansion. This can obviously be understood in terms of a cosmological constant of the right size. Expressed in the plane (Ω_M, Ω_Λ), observational data singles out the region: $\Omega_M \sim$ 0.2 to 0.3 and $\Omega_\Lambda \sim$ 0.7 to 0.8 (FIGURE).

One may stress that, in the hypothesis that this acceleration is due to the cosmological constant, its value is as large as the upper bounds obtained in the previous sections allow:

$$\lambda \sim H_0^2 \ , \quad \ell_\Lambda \sim \ell_{H_0} \ , \quad \Lambda \sim 10^{-3} \text{ eV} \ . \tag{1.44}$$

Regarding the latter scale Λ, which characterizes the vacuum energy ($\rho_{\text{vac}} \equiv \Lambda^4$), one may note the interesting numerical coincidence:

$$\frac{\hbar c}{\Lambda} \sim \sqrt{\ell_{H_0} \ell_P} \sim 10^{-4} \text{ m} \ . \tag{1.45}$$

This relation underlines the fact that the vacuum energy problem involves some deep connection between the infrared regime (the infrared cut-off being ℓ_{H_0}) and the ultraviolet regime (the ultraviolet cut-off being ℓ_P), between the infinitely large and the infinitely small.

1.6 Why now?

In the case where the acceleration of the expansion is explained by a cosmological constant, one has to explain why this constant contribution appears

which allows to cancel the constant vacuum energy at the expense of generating a mass $m_{3/2}$ for the gravitino field (see e.g. [6], section 6.3 for a more complete treatment).

to start to dominate precisely now. This is the "Why now?" or cosmic coincidence problem summarized in Figure 1.1. In order to avoid any reference to us (and hence any anthropic interpretation, see Chapter 5), we may rephrase the problem as follows. Why does the dark energy starts to dominate at a time t_Λ (redshift $z_\Lambda \sim 1$) which almost coincides with the epoch t_G (redshift $z_G \sim 3$ to 5) of galaxy formation?

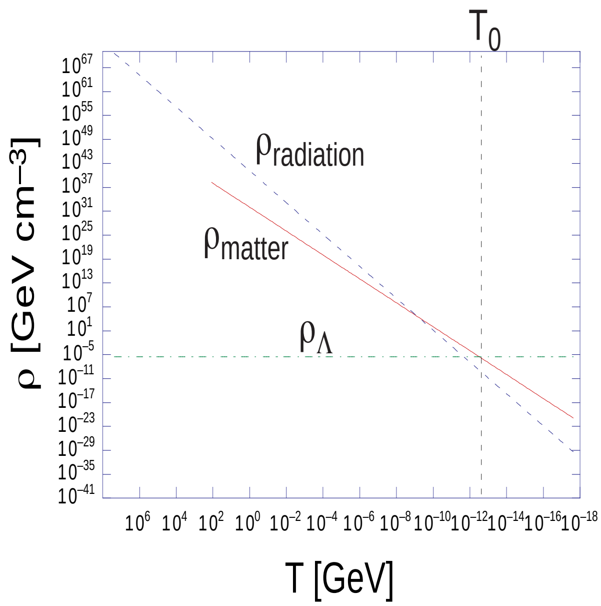


Figure 1.1: The cosmic coincidence problem illustrated in the case of a cosmological constant

An alternate possibility is that the cosmological constant is much smaller or even vanishing and that the acceleration is due to some new form of energy –known as dark energy– or some modifications of gravity. These two possibilities correspond to modifications of either sides of Einstein's equations (1.1). We envisage these two cases in the forthcoming chapters.

Chapter 2

Dark energy

In this section, we assume that some unknown mechanism relaxes the vacuum energy to zero or to a very small value and we introduce some new dynamical component which accounts for the present observation regarding a late acceleration of the universe. We thus try to identify a new component of the energy density with negative pressure:

$$p = w\rho, \quad w < 0. \tag{2.1}$$

Experimental data may constrain such a dynamical component, referred to in the literature as dark energy, just as it did with the cosmological constant. For example, in a spatially flat Universe with only matter and an unknown component X with equation of state $p_X = w_X \rho_X$, one obtains from (1.26) of Chapter 1 with $\rho = \rho_M + \rho_X$, $p = w_X \rho_X$ the following form for the deceleration parameter (compare with (1.28) of Chapter 1)

$$q_0 = \frac{\Omega_M}{2} + (1 + 3w_X)\frac{\Omega_X}{2}, \tag{2.2}$$

where $\Omega_X = \rho_X/\rho_c$. The acceleration of the expansion observed requires that Ω_X dominates with $w_X < -1/3$. Detailed observational results give a more precise constraint on the parameter w_X.

One may easily obtain the time of the onset of the acceleration phase. Indeed, just as in (1.29), one may write the deceleration parameter in terms of redshift:

$$q = \frac{H_0^2}{2H(z)^2}\left[\Omega_{_M}(1+z)^3 + \Omega_{_X}(1+3w_{_X})(1+z)^{3(1+w_X)}\right] \quad . \tag{2.3}$$

This shows that the universe starts accelerating at a redshift value $z_{\rm acc}$ given by

$$1+z_{\rm acc} = \left[-(1+3w_{_X})\frac{\Omega_{_X}}{\Omega_{_M}}\right]^{-1/(3w_{_X})} \quad . \tag{2.4}$$

Setting $\Omega_{_X} \sim 1-\Omega_{_M}$ allows to determine $z_{\rm acc}$ in terms of $\Omega_{_M}$: for $\Omega_{_M} \sim 0.3$ and $w_{_X} \sim -1$, we have $z_{\rm acc} \sim 0.6$.

Another important property of dark energy is that it does not appear to be clustered (just as a cosmological constant). Otherwise, its effects would have been detected locally, as for the case of dark matter.

A central problem that models of dark energy have to address is the following: since matter and dark energy evolve differently, why should they be of the same order at present times? This is the cosmic coincidence problem that we have already discussed in Chapter 1 (see Figure 1.1 for the case of a cosmological constant $w_X = -1$). We will see in the next Chapter how the different dark energy models tackle this issue.

2.1 Scalar fields

A particularly interesting candidate for dark energy in the context of fundamental theories is a scalar[1] field ϕ slowly evolving in its potential $V(\phi)$. More explicitly, we consider the following action

$$\mathcal{S} = \int d^4x\sqrt{-g}\left[-\frac{m_{_P}^2}{2}R + \frac{1}{2}\partial^\mu\phi\partial_\mu\phi - V(\phi)\right] , \tag{2.5}$$

which describes a real scalar field ϕ minimally coupled with gravity. Computing the corresponding energy-momentum tensor, we obtain the energy density and pressure

$$\rho_\phi \;=\; \frac{1}{2}\dot{\phi}^2 + V(\phi) \quad , \tag{2.6}$$

$$p_\phi \;=\; \frac{1}{2}\dot{\phi}^2 - V(\phi) \quad . \tag{2.7}$$

The corresponding equation of motion is, if one neglects the spatial curvature ($k \sim 0$),

$$\ddot{\phi} + 3H\dot{\phi} = -\frac{dV}{d\phi} \ , \tag{2.8}$$

[1]A vector field or any field which is not a Lorentz scalar must have settled down to a vanishing value. Otherwise, Lorentz invariance would be spontaneously broken.

from which we deduce as expected

$$\dot{\rho}_\phi = -3H(p_\phi + \rho_\phi) \ . \tag{2.9}$$

We have for the equation of state parameter

$$w_\phi \equiv \frac{p_\phi}{\rho_\phi} = \frac{\frac{1}{2}\dot{\phi}^2 - V(\phi)}{\frac{1}{2}\dot{\phi}^2 + V(\phi)} \geq -1 \quad . \tag{2.10}$$

If the kinetic energy is subdominant ($\dot{\phi}^2/2 \ll V(\phi)$), we clearly obtain $-1 \leq w_\phi \leq 0$. In any case $-1 \leq w_\phi \leq +1$.

An interesting quantity in the discussion of perturbations associated with this field is the speed of sound[2]

$$c_s^2 \equiv \frac{\delta p}{\delta \rho} \tag{2.13}$$

It is a measure of how the pressure of the field resists gravitation clustering. In most models of dark energy, we have $c_s^2 \sim 1$, which explains why such scalar dark energy does not cluster: its own pressure resists gravitational collapse.

[2] One also finds in the literature the following quantity

$$c_a^2 \equiv \frac{\dot{p}}{\dot{\rho}} \tag{2.11}$$

which defines the adiabatic speed of sound. According to [4], $c_s^2 \neq c_a^2$ in imperfect fluids where dissipative processes generate entropy perturbations (the case of most scalar field models).

Note that, since $\dot{\rho}_\phi = \dot{\phi}\left(\ddot{\phi} + dV/d\phi\right)$ and $\dot{p}_\phi = \dot{\phi}\left(\ddot{\phi} - dV/d\phi\right)$, we have

$$c_a^2 = \frac{\ddot{\phi} - dV/d\phi}{\ddot{\phi} + dV/d\phi} = 1 + 2\frac{dV/d\phi}{3H\dot{\phi}} \ , \tag{2.12}$$

where we have used (2.8). We note that, for a potential slopping down to zero at infinity, the second term is negative ($dV/d\phi < 0$ and $\dot{\phi} > 0$). But the second term is -2 in the case of inflation; hence $c_a^2 < 0$.

2.2 Scaling solutions

One is often looking for *scaling solutions* which we define here[3] as solutions where the ϕ energy density scales as a power of the cosmic scale factor:

$$\rho_\phi \propto a^{-n_\phi} \ , \quad n_\phi \text{ cst} \ . \tag{2.14}$$

Then $\dot{\rho}_\phi/\rho_\phi = -n_\phi H$. In this case, using (2.9), one obtains

$$w_\phi = \frac{n_\phi}{3} - 1 \ . \tag{2.15}$$

Hence the equation of state parameter needs to be constant.

We now consider the evolution of a scalar field ϕ with constant parameter (2.15), during a phase dominated by a background fluid with

$$w_B = \frac{n_B}{3} - 1 \tag{2.16}$$

Following (1.18), we have $a(t) \sim t^{2/n_B}$ ($n_B = 4$ for radiation, 3 for non-relativistic matter,...).

From (2.6) and (2.7), we obtain

$$\dot{\phi}^2 = \frac{n_\phi}{3}\rho_\phi \ , \quad V(\phi) = \left(1 - \frac{n_\phi}{6}\right)\rho_\phi \ . \tag{2.17}$$

Hence $\dot{\phi}^2 \sim a^{-n_\phi}$. Thus $\dot{\phi} \sim t^{-n_\phi/n_B}$. We thus distinguish two cases:

- $n_\phi = n_B$

  We have

  $$\phi = \phi_0 + \frac{2}{\lambda} \ \ln(t/t_0) \ , \tag{2.18}$$

  with λ constant. Then

  $$V(\phi) \sim \rho_\phi \sim a^{-n_\phi} \sim t^{-2} \sim e^{-\lambda\phi} \ . \tag{2.19}$$

  Hence, we find a scaling behavior for the potential

  $$V(\phi) = V_0 e^{-\lambda\phi} \tag{2.20}$$

[3] Beware that some authors use a different definition [e.g. [14]].

in a background such that $n_B = n_\phi$ ($w_B = w_\phi$). The solution of the equation of motion (2.8) then reads

$$\phi = \frac{1}{\lambda} \ln \left(\frac{V_0 \lambda^2}{2} \frac{n_B}{6 - n_B} t^2 \right) \ , \tag{2.21}$$

and the energy density (2.6)

$$\rho_\phi = \frac{12}{\lambda^2 n_B t^2} \ . \tag{2.22}$$

Since $H^2 = (\rho_B + \rho_\phi)/3 \sim [2/(n_B t)]^2$,

$$\frac{\rho_\phi}{\rho_B + \rho_\phi} \sim \frac{n_B}{\lambda^2} \ . \tag{2.23}$$

Hence ρ_ϕ/ρ_B tends to be constant in this scenario. We call this property "tracking". This is obviously compatible with our initial assumptions only if $\lambda^2 > n_B$.

What happens if $\lambda^2 \leq n_B$?

It turns out that the scaling solution corresponds to a totally different regime: the scalar field is the dominant contribution to the energy density. We do not have to redo the calculation: it is identical to the previous one with the only changes $w_B \to w_\phi$ or $n_B \to n_\phi$ (for example, $H^2 = (\rho_B + \rho_\phi)/3 \sim [2/(n_\phi t)]^2$: the scalar energy density determines the evolution of the Universe). But then (2.23) reads $1 \sim n_\phi/\lambda^2$, i.e.

$$w_\phi = -1 + \frac{\lambda^2}{3} \ . \tag{2.24}$$

Thus, if $\lambda^2 < 2$, the scalar field ϕ may provide the dark energy component.

To summarize the two regimes that we have obtained for the exponential potential (2.20):

– if $\lambda^2 \leq n_B$, the scaling solution has $w_\phi = -1 + \lambda^2/3$ and $\rho_\phi/(\rho_\phi + \rho_B) \sim 1$ (ϕ is the dominant species),

– if $\lambda^2 > n_B$, the scaling solution has $w_\phi = w_B$ and $\rho_\phi/(\rho_B + \rho_\phi) \sim n_B/\lambda^2$ (the background energy density dominates; the scalar field energy density tracks it).

- $n_\phi \neq n_B$

  Then $\phi \sim t^{-\frac{n_\phi}{n_B}+1}$ and we now have

$$V(\phi) \sim \rho_\phi \sim a^{-n_\phi} \sim t^{-2n_\phi/n_B} \sim \phi^{-2\frac{n_\phi}{n_B - n_\phi}} \ . \tag{2.25}$$

  Hence, we find a scaling behaviour for the potential, known as the Ratra-Peebles potential [26, 24],

$$V(\phi) = \frac{M^{4+\alpha}}{\phi^\alpha} \ , \quad \alpha > 0 \ , \tag{2.26}$$

  in a background characterized by $n_B \neq n_\phi$ (or $w_B \neq w_\phi$). We have

$$n_\phi = \frac{\alpha n_B}{\alpha + 2} \quad \text{or} \quad w_\phi = \frac{\alpha w_B - 2}{\alpha + 2} \ . \tag{2.27}$$

  The complete solution of the equation of motion (2.8) is

$$\phi = \left(\frac{\alpha(\alpha+2)^2 n_B \ M^{4+\alpha} t^2}{2\left[6(\alpha+2) - n_B \alpha\right]} \right)^{\frac{1}{\alpha+2}} \ . \tag{2.28}$$

It turns out that these scaling solutions correspond to *attractors* in the cosmological evolution of the scalar field. We will take the example of the exponential potential (2.20). A small perturbation $\delta\phi$ satisfies the equation[4] $\delta\ddot{\phi} + 3H\delta\dot{\phi} + \lambda^2 V_0 e^{-\lambda\phi}\delta\phi = 0$, which we may write, using (2.21),

$$\delta\ddot{\phi} + \frac{6}{n_B t}\delta\dot{\phi} + 2\frac{6 - n_B}{n_B t^2}\delta\phi = 0 \ . \tag{2.29}$$

The solution is $\delta\phi \sim t^\gamma$ with

$$\gamma = \frac{-(6 - n_B) \pm \sqrt{-3(6 - n_B)(3n_B - 2)}}{2n_B} \tag{2.30}$$

The term under the square root is negative for standard values of n_B ($2/3 < n_B < 6$). The square root thus contributes as an oscillating term and the

[4] We assume that the evolution of the scale factor is set by the background: $H \sim 2/(n_B t)$ does not depend on ϕ to a first approximation. As we saw, this imposes $\lambda^2 \gg n_B$.

two solutions (2.30) decay as $\delta\phi \sim t^{-(6-n_B)/2n_B}$: the solution (2.21) is an attractor.

Exercise 2-1 : We show in this exercise that the solution (2.28) of the Ratra-Peebles potential (2.26) is also an attractor.

a) Show that a small perturbation $\delta\phi$ satisfies the equation

$$\delta\ddot{\phi} + \frac{6}{n_B t}\delta\dot{\phi} + \frac{2(\alpha+1)}{n_B(\alpha+2)^2 t^2}\left[6(\alpha+2) - n_B\alpha\right]\delta\phi = 0 \ . \tag{2.31}$$

b) Look for a solution of the form $\delta\phi \sim t^\gamma$ and express γ in terms of n_B and α.

c) Assume $\alpha > 0$ and standard values of n_B. Show that the two solutions obtained in b) decay as $\delta\phi \sim t^{-(6-n_B)/2n_B}$: the solution (2.28) is an attractor.

Hints: b)

$$\gamma = -\frac{6-n_B}{2n_B} \pm \frac{\sqrt{-\left[3\alpha^2(3n_B-2)(6-n_B) - 12\alpha(n_B^2 - 16n_B + 12) - 4(n_B^2 - 36n_B + 36)\right]}}{2n_B(\alpha+2)} \ . \tag{2.32}$$

c) The reduced discriminant of the second order polynomial in α which is under the square root is simply $288 n_B^3 > 0$. The corresponding roots are then negative for $8 - \sqrt{52} \sim 0.79 < n_B < 6$ and the term is thus negative for $\alpha > 0$ and n_B in this range. The square root contributes as an oscillating term and the two solutions corresponding to (2.32) decay as $\delta\phi \sim t^{-(6-n_B)/2n_B}$.

2.3 Slow roll

Let us consider the case of the Ratra-Peebles potential (2.26).

We have found above the attractor scaling solution [26, 24] $\phi \propto a^{n_B/(2+\alpha)}$, $\rho_\phi \propto a^{-\alpha n_B/(2+\alpha)}$ in the case where the background density dominates. Thus ρ_ϕ decreases at a slower rate than the background density ($\rho_B \propto a^{-n_B}$) and tracks it until it becomes of the same order, at a given value a_Q. We thus have:

$$\frac{\phi}{m_P} \sim \left(\frac{a}{a_Q}\right)^{n_B/(2+\alpha)} , \tag{2.33}$$

$$\frac{\rho_\phi}{\rho_B} \sim \left(\frac{a}{a_Q}\right)^{2n_B/(2+\alpha)} . \tag{2.34}$$

Exercise 2-2 : Compute the time t_Q at which $\rho_\phi \sim \rho_Q$ in terms of M and $m_{_P}$. Check that ϕ at t_Q does not depend on M.

Hints: $\rho_M \sim m_{_P}^2/t^2$ and $\rho_\phi \sim M^{\frac{2(\alpha+4)}{\alpha+2}} t^{-\frac{2\alpha}{\alpha+2}}$ give $t_Q \sim m_P^{\frac{\alpha+2}{2}} M^{-\frac{\alpha+4}{2}}$.

The corresponding value for the equation of state parameter is given by (2.27):

$$w_\phi = -1 + \frac{\alpha(1+w_B)}{2+\alpha}. \tag{2.35}$$

Shortly after ϕ has reached for $a = a_Q$ a value of order m_P, it satisfies the standard slow roll conditions

$$\epsilon \equiv \frac{1}{2}\left(\frac{m_{_P} V'}{V}\right)^2 = (\alpha/2)(m_{_P}/\phi)^2 \ll 1\,, \quad \eta \equiv \frac{m_{_P}^2 V''}{V} = \alpha(\alpha+1)(m_{_P}/\phi)^2 \ll 1\,. \tag{2.36}$$

Therefore (2.35) provides a good approximation to the present value of w_ϕ. Thus, at the end of the matter-dominated era, this field may provide the quintessence component that we are looking for.

Two features are interesting in this respect. One is that this scaling solution is reached for rather general initial conditions, *i.e.* whether ρ_ϕ starts of the same order or much smaller than the background energy density [34].

The second is the present value of ρ. Typically, since in this scenario ϕ is of order $m_{_P}$ when the quintessence component emerges, we must choose the scale M in such a way that $V(m_{_P}) \sim \rho_c$. The constraint reads:

$$M \sim \left(H_0^2 m_{_P}^{2+\alpha}\right)^{1/(4+\alpha)}. \tag{2.37}$$

We may note that this gives for $\alpha = 2$, $M \sim 10$ MeV, not such an atypical scale for high energy physics.

Exercise 2-3 : In the case of slow roll, the equation of motion (2.8) simply reads $3H\dot{\phi} = -V'(\phi)$.

a) Under this assumption, show that

$$\ddot{\phi} = -\frac{4\pi G_{_N}}{3}\frac{V'}{H^2}\sum_i (p_i + \rho_i) \tag{2.38}$$

where the summation is over all components of the Universe.

b) Deduce that, in the case where only matter and dark energy are non-negligible at present time t_0 ($\Omega_M + \Omega_\phi = 1$),

$$\left.\frac{\ddot{\phi}}{V'}\right|_{t_0} \sim -\frac{1}{2}(1-\Omega_\phi) \ . \tag{2.39}$$

Hence slowroll requires that $\Omega_\phi \sim 1$.

Hints: a) Use $\dot{H} = -4\pi G_{_N} \sum_i (p_i + \rho_i)$.

2.4 Quintessential problems

However appealing, the quintessence idea is difficult to implement in the context of realistic models [12, 22]. The main problem lies in the fact that the quintessence field must be extremely weakly coupled to ordinary matter. This problem can take several forms:

• the quintessence field must be very light. If we return to our example of Ratra-Peebles potential in (2.26), $V''(m_{_P})$ provides an order of magnitude for the mass-squared of the quintessence component:

$$m_\phi \sim M \left(\frac{M}{m_{_P}}\right)^{1+\alpha/2} \sim H_0 \sim 10^{-33} \text{ eV}. \tag{2.40}$$

using (2.37). This might argue for a pseudo-Goldstone boson nature of the scalar field that plays the rôle of quintessence. This field must in any case be very weakly coupled to matter; otherwise its exchange would generate observable long range forces. Eötvös-type experiments put very severe constraints on such couplings.

• because the *vev* of ϕ is of order $m_{_P}$, one must take into account all non-renormalisable interactions of order $(\phi/m_{_P})^n$. For example, in a supersymmetric context, the full supergravity corrections must be included. One may then argue [9] that this could put in jeopardy the positive definiteness of the scalar potential, a key property of the quintessence potential. This may point towards supergravity models where $\langle W \rangle = 0$ (but not its derivatives) or to no-scale type models: in the latter case, the presence of 3 moduli fields T^i with Kähler potential $K = -\sum_i \ln(T^i + \bar{T}^i)$ cancels the negative contribution $-3|W|^2$ in the supergravity potential.

• it is difficult to find a symmetry that would prevent any coupling of the form $\beta(\phi/m_P)^n F^{\mu\nu} F_{\mu\nu}$ to the gauge field kinetic term. Since the quintessence behavior is associated with time-dependent values of the field of order m_P, this would generate, in the absence of fine tuning, corrections of order one to the gauge coupling. But the time dependence of the fine structure constant for example is very strongly constrained: $|\dot{\alpha}/\alpha| < 5 \times 10^{-17}\mathrm{yr}^{-1}$. This yields a limit [12]:

$$|\beta| \leq 10^{-6} \frac{m_P H_0}{< \dot{\phi} >}, \tag{2.41}$$

where $< \dot{\phi} >$ is the average over the last 2×10^9 years.

All the preceding shows that there is extreme fine tuning in the couplings of the quintessence field to matter, unless they are forbidden by some symmetry. This is somewhat reminiscent of the fine tuning associated with the cosmological constant. In fact, *the quintessence solution does not claim to solve the cosmological constant (vacuum energy) problem* described above. If we take the example of a supersymmetric theory, the dynamical cosmological constant provided by the quintessence component clearly does not provide enough amount of supersymmetry breaking to account for the mass difference between scalars (sfermions) and fermions (quarks and leptons): at least 100 GeV. There must be other sources of supersymmetry breaking and one must fine tune the parameters of the theory in order not to generate a vacuum energy that would completely drown ρ_ϕ.

However, the quintessence solution shows that, once this fundamental problem is solved, one can find explicit fundamental models that effectively provide the small amount of cosmological constant that seems required by experimental data.

Chapter 3

Dark energy scenarios

3.1 Runaway quintessence

Particle physics models, in particular in the context of supersymmetry or string theory, provide numerous models of dark energy scalar fields. We present some of them in this Section.We start with models illustrating the two types of potentials that we have singled out so far: the exponential potential and the Ratra-Peebles potential. Historically, they correspond to the first models studied, already back in 1988 [32, 24, 26].

A runaway potential is frequently present in models where supersymmetry is dynamically broken. Supersymmetric theories are characterized by a scalar potential with many flat directions, *i.e.* directions ϕ in field space for which the potential vanishes. The corresponding degeneracy is lifted through dynamical supersymmetry breaking. In some instances (dilaton or compactification radius), the field expectation value $\langle\phi\rangle$ actually provides the value of the strong interaction coupling. Then at infinite ϕ value, the coupling effectively goes to zero together with the supersymmetry breaking effects and the flat direction is restored: the potential decreases monotonically to zero as ϕ goes to infinity.

Let us take the example of supersymmetry breaking by gaugino condensation in effective superstring theories. The value g_0 of the gauge coupling at the string scale M_S is provided by the vacuum expectation value of the dilaton field s (taken to be dimensionless by dividing by m_P) present among the massless string modes: $g_0^2 = \langle s\rangle^{-1}$. If the gauge group has a one loop beta function coefficient $b > 0$, then the running gauge coupling becomes strong at the scale

$$\Lambda \sim M_S e^{-8\pi^2/(bg_0^2)} = M_S e^{-8\pi^2 s/b} \ . \tag{3.1}$$

At this scale, the gaugino fields are expected to condense. Through dimensional analysis, the gaugino condensate $< \bar{\lambda}\lambda >$ is expected to be of order Λ^3. Terms quadratic in the gaugino fields thus yield in the effective theory below condensation scale a potential for the dilaton:

$$V \sim \left| < \bar{\lambda}\lambda > \right|^2 \propto e^{-48\pi^2 s/b}. \tag{3.2}$$

The s-dependence of the potential is of course more complicated and one usually looks for stable minima with vanishing cosmological constant. But the behavior (3.1) is characteristic of the large s region and provides a potential slopping down to zero at infinity as required in the quintessence approach. A similar behavior is observed for moduli fields whose *vev* describes the radius of the compact manifolds which appear from the compactification from 10 or 11 dimensions to 4 in superstring theories.

Let us take therefore the example of an exponentially decreasing potential. and the self-interactions of which are described by the potential:

$$V(\phi) = V_0 e^{-\lambda\phi/m_P}, \tag{3.3}$$

where V_0 is a positive constant.

The semi-realistic models discussed earlier tend to give large values of λ and thus the tracking solution as an attractor ($w_\phi = w_B$). For example, in the case (3.2) where the scalar field is the dilaton, $\lambda = 48\pi^2/b$ with $b = 90$ for a E_8 gauge symmetry down to $b = 9$ for $SU(3)$. Moreover [16], on the observational side, the condition that ρ_ϕ should be subdominant during nucleosynthesis (in the radiation-dominated era) imposes to take rather large values of λ. Typically requiring $\rho_\phi/(\rho_\phi + \rho_B)$ to be then smaller than 0.2 imposes $\lambda^2 > 20$.

Models of dynamical supersymmetry breaking easily provide a potential of the Ratra-Peebles type discussed above [5]. Let us consider supersymmetric QCD with gauge group $SU(N_c)$ and $N_f < N_c$ flavors, *i.e.* N_f quarks Q_g (resp. antiquarks $\bar{Q}^g$), $g = 1 \cdots N_f$, in the fundamental $\mathbf{N_c}$ (resp. antifundamental $\bar{\mathbf{N}}_\mathbf{c}$) of $SU(N_c)$. At the scale of dynamical symmetry breaking Λ where the gauge coupling becomes strong, boundstates of the meson type form: ${M_f}^g = Q_f \bar{Q}^g$. The dynamics is described by a superpotential which can be computed non-perturbatively using standard methods:

$$W = (N_c - N_f) \frac{\Lambda^{(3N_c - N_f)/(N_c - N_f)}}{(\det\ M)^{1/(N_c - N_f)}} \ . \tag{3.4}$$

Such a superpotential has been used in the past but with the addition of a mass or interaction term (*i.e.* a positive power of M) in order to stabilize the condensate. One does not wish to do that here if M is to be interpreted as a runaway quintessence component. For illustration purpose, let us consider a condensate diagonal in flavor space: ${M_f}^g \equiv \phi^2 \delta_f^g$. Then the potential for ϕ has the form (2.9), with $\alpha = 2(N_c + N_f)/(N_c - N_f)$. Thus,

$$w_\phi = -1 + \frac{N_c + N_f}{2N_c}(1 + w_B), \tag{3.5}$$

which clearly indicates that the meson condensate is a potential candidate for a quintessence component.

3.2 k-essence and nontrivial structure for the kinetic terms

In order to introduce this class of models, we will take our inspiration from the description of a relativistic particle in special relativity. From the Lagrangian $L = -m\sqrt{1 - \dot{q}^2}$, where m is the mass of the particle and $q(t)$ its one-dimensional position, one derives its energy $E = m/\sqrt{1 - \dot{q}^2}$ and momentum $k = m\dot{q}/\sqrt{1 - \dot{q}^2}$. Moving to field theory, one may replace $q(t)$ by $\phi(\mathbf{x}, t)$, $\dot{q}^2$ by $\partial^\mu \phi \partial_\mu \phi$ (which would simply read $\dot{\phi}^2$ in the case of a homogeneous field) and even make the mass m a field-dependent function $\mu(\phi)$. The corresponding Lagrangian density is thus

$$\mathcal{L} = -\mu(\phi)\sqrt{1 - \partial^\mu \phi \partial_\mu \phi} \ . \tag{3.6}$$

Such non-trivial structure in the kinetic terms of scalar fields often appears in the context of string and brane theory. For example, non-BPS Dp-branes suffer from an instability under which all open string states disappear: a tachyonic mode is present and the system should relax to the minimum of the tachyonic potential [27, 28]. This is described at the level of the effective field theory by a Dirac-Born-Infeld (DBI) action

$$\mathcal{S} = \int d^{p+1}x \; V(\phi)\sqrt{-\det\left[g_{mn} + 2\pi\alpha' F_{mn} + \partial_m\phi\partial_n\phi\right]} \;, \tag{3.7}$$

where ϕ is the tachyon field, $V(\phi)$ the tachyon potential and $\alpha' \equiv M_{_S}^{-2}$ the string constant. Disregarding the gauge field, we obtain (see Exercise 3-1)

$$\mathcal{S} = \int d^{p+1}x \, V(\phi)\sqrt{-g}\sqrt{\det\left[\delta^m_n + g^{mr}\partial_r\phi\partial_n\phi\right]} = \int d^{p+1}x \, V(\phi)\sqrt{-g}\sqrt{1 + g^{mn}\partial_m\phi\partial_n\phi}\,, \tag{3.8}$$

where $g = \det(g_{mn})$. The same action provides the effective description of a Dp-brane anti-Dp-brane system: ϕ then describes the distance between the two branes.

Exercise 3-1 : Show that $\det\left(\delta^m_n + A^m B_n\right) = 1 + A^m B_m$.

More recently, a similar system has been considered: a probe $D3$-brane travelling down a five-dimensional *warped* throat geometry. The warping means that the $d+1 = 4$-dimensional metric on the brane is ϕ-dependent, namely $f(\phi)^{-1})g_{\mu\nu}$. Thus, assuming a constant potential, one obtains [29, 1]

$$\mathcal{S} = \int d^4x \; \sqrt{-g} f^{-2}(\phi)\sqrt{1 + f(\phi)g^{\mu\nu}\partial_\mu\phi\partial_\nu\phi} \;. \tag{3.9}$$

In the case of throats coming from IIB flux compactifications [1],

$$f(\phi) \sim \frac{\lambda}{\phi^4} \;, \tag{3.10}$$

and an additive potential term arises from the couplings of the D-brane to background RR fluxes.

Models with non-trivial kinetic terms have been proposed to account for dark energy [2]: *k-essence* models are based on the following generic action

$$\mathcal{S} = \int d^4x\sqrt{-g}\left[-\frac{m_{_P}^2}{2}R + \mathcal{L}(X,\phi)\right] \;, \quad \text{where} \quad X \equiv \frac{1}{2}g^{\mu\nu}\partial_\mu\phi\partial_\nu\phi \;. \tag{3.11}$$

Variation of the action with respect to ϕ yields the energy-momentum for the scalar field:[1]

$$T_{\mu\nu} \equiv \frac{2}{\sqrt{-g}} \frac{\delta \mathcal{S}}{\delta g^{\mu\nu}} = \mathcal{L}_{,X} \partial_\mu \phi \partial_\nu \phi - g_{\mu\nu} \mathcal{L} \ . \tag{3.12}$$

This has a hydrodynamic description. Indeed, introducing

$$U_\mu \equiv \frac{\partial_\mu \phi}{\sqrt{2X}} \ , \tag{3.13}$$

$T_{\mu\nu}$ has the perfect fluid form (2.4) with

$$p(X, \phi) \equiv \mathcal{L}(X, \phi) \ , \quad \rho(X, \phi) \equiv 2X \mathcal{L}_{,X} - \mathcal{L} \ . \tag{3.14}$$

The sound speed can be expressed as [19]

$$c_s^2 = \frac{p_{,X}}{\rho_{,X}} = \frac{\mathcal{L}_{,X}}{\mathcal{L}_{,X} + 2X \mathcal{L}_{,XX}} \ . \tag{3.15}$$

In the case where $\mathcal{L}$ depends only on X, then one has $p = p(\rho)$ and one recovers the usual $c_s^2 = \partial p / \partial \rho$.

The equation of motion for the scalar field reads

$$\tilde{G}^{\mu\nu} \partial_\mu \phi \partial_\nu \phi + 2X \mathcal{L}_{,X\phi} - \mathcal{L}_{,\phi} = 0 \ , \tag{3.16}$$

where [3] [2]

$$\tilde{G}^{\mu\nu} \equiv \mathcal{L}_{,X} g^{\mu\nu} + \mathcal{L}_{,XX} \ \partial^\mu \phi \partial^\nu \phi \ . \tag{3.17}$$

Tracking of the kind discussed for quintessence models occurs only in the radiation-dominated era; a new attractor solution where quintessence acts as a cosmological constant is activated by the onset of matter domination.

3.3 Pseudo-Goldstone boson

There exists a class of models [18] very close in spirit to the case of runaway quintessence: they correspond to a situation where a scalar field has not yet reached its stable groundstate and is still evolving in its potential.

[1] The null energy condition $T_{\mu\nu} n^\mu n^\nu \geq 0$ (n^μ null vector: $g_{\mu\nu} n^\mu n^\nu = 0$) is satisfied if $\mathcal{L}_{,X} \geq 0$.

[2] Since det $\tilde{G}_{\mu\nu} = g^{-1} c_s^{-2} \mathcal{L}_{,X}$, this second order differential equation is hyperbolic – i.e. det $\tilde{G}_{\mu\nu} < 0$ – and thus describes the time evolution of the system provided $c_s^2 > 0$.

More specifically, let us consider a potential of the form:

$$V(\phi) = M^4 v\left(\frac{\phi}{f}\right) \ , \tag{3.18}$$

where M is the overall scale, f is the vacuum expectation value $< \phi >$ and the function v is expected to have coefficients of order one. If we want the potential energy of the field (assumed to be close to its *vev* f) to give a substantial fraction of the energy density at present time, we must set

$$M^4 \sim \rho_c \sim H_0^2 m_P^2 \ . \tag{3.19}$$

However, requiring that the evolution of the field ϕ around its minimum has been overdamped by the expansion of the Universe until recently imposes

$$m_\phi^2 = \frac{1}{2} V''(f) \sim \frac{M^4}{f^2} \leq H_0^2. \tag{3.20}$$

Let us note that this is again one of the slowroll conditions familiar to the inflation scenarios.

From (3.19) and (3.20), we conclude that f is of order $m_{_P}$ (as the value of the field ϕ in runaway quintessence) and that $M \sim 10^{-3}$ eV (not surprisingly, this is the scale Λ typical of the cosmological constant). As we have seen, the field ϕ must be very light: $m_\phi \sim h_0 \times 10^{-60} m_P \sim h_0 \times 10^{-33}$ eV. Such a small value is only natural in the context of an approximate symmetry: the field ϕ is then a pseudo-Goldstone boson. A typical example of such a field is provided by the string axion field. In this case, the potential simply reads:

$$V(\phi) = M^4 \left[1 + \cos(\phi/f)\right]. \tag{3.21}$$

3.4 Coupling dark energy with dark matter

3.5 Quintessential inflation

One may note that, in the tracking solution, when ϕ reaches values of order $m_{_P}$, it satisfies the slow roll conditions of an inflation model. The last possibility that I will discuss goes in this direction one step further. It is known under several names: deflation [30], kination [21], quintessential inflation [25]. It is based on the remark that, if a field ϕ is to provide a dynamical cosmological constant under the form of quintessence, it is a good candidate to account for an inflationary era where the evolution is dominated by the vacuum energy. In other words, are the quintessence component and the inflaton the same unique field?

In this kind of scenario, inflation (where the energy density of the Universe is dominated by the ϕ field potential energy) is followed by reheating where matter-radiation is created by gravitational coupling during an era where the evolution is driven by the ϕ field kinetic energy (which decreases as a^{-6}). Since matter-radiation energy density is decreasing more slowly, this turns into a radiation-dominated era until the ϕ energy density eventually emerges as in the quintessence scenarios described above.

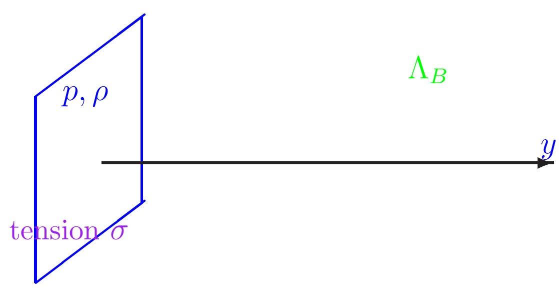
Chapter 4

Modification of gravity

4.1 Extended gravities

4.2 Braneworlds

We start by illustrating brane cosmology on a 5-dimensional toy model which consists of a 3-brane with vacuum energy σ: this provides the tension of the brane. Matter with pressure p and energy density ρ is localized on the brane. Away from the brane, the bulk of 5-dimensional spacetime (from now on referred to as the "bulk") is empty but has vacuum energy Λ_B:



The fact that there is a single dimension perpendicular to the brane allows to solve the system completely. Using the 5-dimensional Einstein equations and the Israel junction conditions which express the discontinuities of the metric coefficients (or more precisely of their derivatives in the fifth direction orthogonal to the brane) due to the localization of matter on the brane, one obtains a generalized Friedmann equation on the brane. This equation provides the evolution of the cosmic scale factor $a_0(t)$ on the brane through the Hubble parameter $H \equiv \dot{a}_0(t)/a_0(t)$ [8, 7, 15, 13]:

$$H^2 = \frac{1}{6M_5^3}\Lambda_B + \frac{1}{36M_5^6}\sigma^2 + \frac{1}{18M_5^6}\sigma\rho + \frac{1}{36M_5^6}\rho^2 + \frac{\mathcal{C}}{a_0^4} - \frac{k}{a_0^2} \tag{4.1}$$

where the fundamental scale M_5 is related to the 5-dimensional Newton's constant by $8\pi G_5 \equiv M_5^{-3}$ and $\mathcal{C}$ is a constant to be discussed below.

This should be compared with the standard Friedmann equation (see (1.16))

$$H^2 = \frac{\lambda}{3} + \frac{1}{3m_P^2}\,\rho - \frac{k}{a^2} \tag{4.2}$$

where λ is the 4-dimensional cosmological constant.
Comparing (4.1) with (4.2), one may remark that the 4-dimensional cosmological constant λ receives contributions both from the bulk vacuum energy Λ_B and from the brane tension σ (squared). This has the advantage of decoupling somewhat the cosmological constant from the brane vacuum energy i.e. the tension σ. Of course, the cosmological constant problem remains: it requires a delicate fine tuning between the brane and the bulk vacuum energies. Various attempts have been made to relieve this fine-tuning through some dynamics in the bulk (self-tuning) but they have not been successful.

4.3 Induced gravity

Chapter 5

Back to the cosmological constant

5.1 Relaxation mechanisms

From the point of view of high energy physics, it is however difficult to imagine a rationale for a pure cosmological constant, especially if it is nonzero but small compared to the typical fundamental scales (electroweak, strong, grand unified or Planck scale). There should be dynamics associated with this form of energy.

For example, in the context of string models, any dimensionful parameter is expressed in terms of the fundamental string scale M_S and of vacuum expectation values of scalar fields. The physics of the cosmological constant would then the physics of the corresponding scalar fields.

Indeed, it was difficult from the start to envisage string theory in the context of a true cosmological constant. The corresponding spacetime is known as de Sitter spacetime and has an event horizon. This is difficult to reconcile with the S-matrix approach of string theory in the context of conformal invariance. More precisely, in the S-matrix approach, states are asymptotically (i.e. at times $t \to \pm\infty$) free and interact only at finite times: the S-matrix element between an incoming set of free states and an outgoing set yields the probability associated with such a transition. In string theory, the states are strings and a diagram such as the one given in Figure 5.1 gives a contribution to the S-matrix element. But conformal invariance, which is a key element, imposes that the string world-sheet can be deformed at will:

this is difficult to reconcile with the presence of a horizon and the requirement of asymptotically free states.

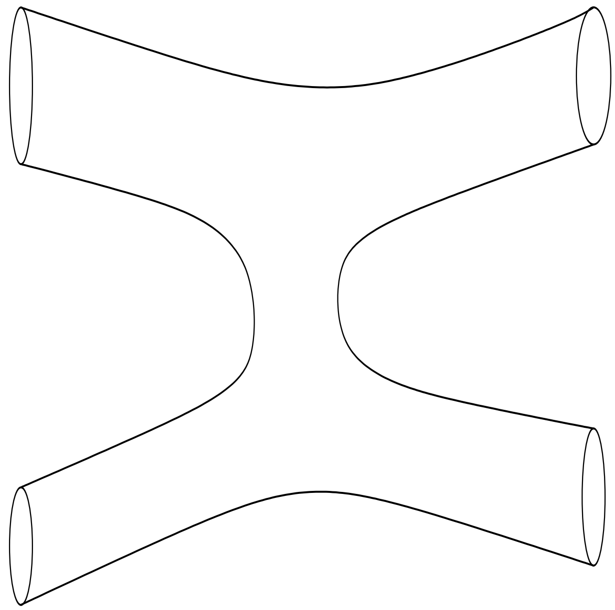


Figure 5.1:

Steven Weinberg [31] has constrained the possible mechanisms for the relaxation of the cosmological constant by proving the following "no-go" theorem: *it is not possible to obtain a vanishing cosmological constant as a consequence of the equations of motion of a finite number of fields.*

Indeed, let us consider N such fields φ_n, $n = 1, \cdots, N$. In the equilibrium configuration these fields are constant and their equations of motion simply read

$$\frac{\delta \mathcal{L}}{\delta \varphi_n} = 0 \quad . \tag{5.1}$$

Remembering that $\lambda_{\text{eff}} \sim \langle T^{\mu}{}_{\mu} \rangle$ where the energy-momentum tensor may be obtained from varying the metric ($T^{\mu\nu} = \delta\mathcal{L}/\delta g_{\mu\nu}$), we see that the vanishing of the cosmological constant is a consequence of the equations (5.1) if we can find N functions $f_n(\varphi)$ such that

$$2g_{\mu\nu}\frac{\delta\mathcal{L}}{\delta g_{\mu\nu}} = \sum_n \frac{\delta\mathcal{L}}{\delta\varphi_n} f_n(\varphi) \quad . \tag{5.2}$$

This amounts to a symmetry condition, the invariance of the Lagrangian $\mathcal{L}$ under

$$\delta g_{\mu\nu} = 2\alpha g_{\mu\nu} \quad , \quad \delta\varphi_n = -\alpha f_n(\varphi) \quad . \tag{5.3}$$

However, one can redefine the fields φ_n, $n = 1, \cdots, N$ into σ_a, $a = 1, \cdots, N-1$ and φ in such a way that the invariance reads

$$\delta g_{\mu\nu} = 2\alpha g_{\mu\nu} \quad , \quad \delta\sigma_a = 0 \quad , \quad \delta\varphi = -\alpha \quad . \tag{5.4}$$

The Lagrangian which satisfies this invariance is written

$$\mathcal{L} = \sqrt{\text{Det } (e^{2\varphi} g_{\mu\nu})}\mathcal{L}_0(\sigma) = e^{4\varphi}\sqrt{|g|}\ \mathcal{L}_0(\sigma) \quad , \tag{5.5}$$

which does not provide a solution to the relaxation of the cosmological constant, as can be seen by redefining the metric: $\hat{g}_{\mu\nu} = e^{2\varphi} g_{\mu\nu}$ (in the new metric, the field φ has only derivative couplings).

Obviously, Weinberg's no-go theorem relies on a series of assumptions: Lorentz invariance, *finite* number of *constant* fields, possibility of globally redefining these fields...

5.2 Fluxes and the landscape

An inspiring example is provided by the Brown-Teitelboim mechanism [10, 11] where the quantum creation of closed membranes leads to a reduction of the vacuum energy inside. This is easier to understand on a toy model with a single spatial dimension.

Let us thus consider a line and establish along it a constant electric field $E_0 > 0$: the corresponding (vacuum) energy is $E_0^2/2$. Quantum creation of a pair of $\pm q$-charged particles ($q > 0$) leads to the formation of a region (between the two charges) where the electric field is partially screened to the value $E_0 - q$ and thus the vacuum energy is decreased to the value $(E_0 - q)^2/2$. Quantum creation of pairs in the new region will subsequently decrease the value of the vacuum energy. The process ends in flat space when the electric field reaches the value $E \leq q/2$ because it then becomes insufficient to separate the pairs created.

In a truly three-dimensional universe, the quantum creation of pairs is replaced by the quantum creation of membranes and the one-dimensional electric field is replaced by a tensor field $A_{\mu\nu\rho}$.There are two potential problems with such a relaxation of the cosmological constant.

First, since the region of small cosmological constant originates from regions with large vacuum energies, hence exponential expansion, it is virtually empty: matter has to be produced through some mechanism yet to be specified. The second problem has to do with the multiplicity of regions with different vacuum energies: why should we be in the region with the smallest value? Such questions are crying for an anthropic type of answer: some regions of spacetime are preferred because they allow the existence of observers.

5.3 Anthropic considerations

The anthropic principle approach can be sketched as follows. We consider regions of spacetime with different values of t_G (time of galaxy formation) and t_Λ, the time when the cosmological constant starts to dominate i.e. when the Universe enters a de Sitter phase of exponential expansion. Clearly galaxy formation must precede this phase otherwise no observer (similar to us) would be able to witness it. Thus $t_G \leq t_\Lambda$. On the other hand, regions with $t_\Lambda \gg t_G$ have not yet undergone any de Sitter phase of reacceleration and are thus "phase-space suppressed" compared with regions with $t_\Lambda \sim t_G$. Hence the regions favoured have $t_\Lambda \gtrsim t_G$ and thus $\rho_\Lambda \sim \rho_M$.

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