4 Static HII regions

- **HII regions** are regions of the ISM in which the hydrogen is highly ionized (hence HII), and where this ionization is due to the effects of photoionization from a nearby O or B-type star. Note: all HII regions consist of ionized hydrogen, but not all clouds of ionized hydrogen are HII regions – a good counter-example is the gas within a supernova remnant, which is highly ionized, but which would not typically be referred to as being part of an HII region.

- Understanding how HII regions form and evolve is important for several different reasons. Firstly, ionizing radiation from massive stars represents one of the main forms of **stellar feedback** on the ISM, and hence it is important to understand the effects of this feedback. Secondly, the study of nearby HII regions, that can be well-resolved observationally, can tell us a great deal about the gas distribution in regions where massive stars are forming, complementing what we can learn from observations of the colder atomic and molecular phases. Finally, spectral line and radio continuum emission from HII regions are two of the main tracers of star formation in other galaxies, and so it is important to understand where this emission comes from.

4.1 Photoionization equilibrium

4.1.1 The simplest case: pure hydrogen

- Our study of the physics of HII regions begins with the simplest possible case: a gas of pure hydrogen which is in photoionization equilibrium – i.e. where each photoionization is balanced by a recombination. An any point within such an HII region, the following equation holds:

$$n_{\rm H} \int_{\nu_0}^{\infty} \frac{4\pi J_{\nu}}{h\nu} \sigma_{\nu}({\rm H}) {\rm d}\nu = n_{\rm H^+} n_{\rm e^-} \alpha. \tag{1}$$

  Here, $n_{\rm H}$, $n_{\rm H^+}$ and $n_{\rm e^-}$ are the number densities of H, $\rm H^+$ and electrons, respectively, J_ν is the mean specific intensity of the radiation field, $\sigma_\nu({\rm H})$ is the photoionization cross-section of atomic hydrogen and α is the recombination coefficient.

- Breaking this down a bit: J_ν is the energy per unit area per unit time per unit solid angle per unit frequency interval, so the combination $4\pi J_\nu / h\nu$ gives the **number of photons** per unit area per unit time per unit frequency. By multiplying by σ_ν and integrating over frequency, we can convert this to the number of ionizations per hydrogen atom per unit time. Finally, multiplying by the number density of atomic hydrogen gives the **photoionization rate** per unit volume, which in equilibrium must balance the recombination rate per unit volume.

- We can use this expression to demonstrate that the gas within an HII region must be very highly ionized. Consider, for instance a bright O star that emits a total of

$N_{\rm ion} = 10^{49}$ ionizing photons per second. At a distance $R = 5\,{\rm pc}$ from such a star, the ionizing photon flux is

$$4\pi J_{\rm ion} = \frac{N_{\rm ion}}{4\pi R^2 h\nu} \simeq 3.3 \times 10^9 \,{\rm photons\,s^{-1}}, \quad (2)$$

where for the moment we have ignored any absorption of ionizing photons in the gas between the star and this point.

- If all of these photons have energies close to 13.6 eV, then we can approximate σ_ν with its value at the hydrogen ionization threshold, $\sigma_0 = 6.3 \times 10^{-18}$. If we do so, we find that the ionization rate per unit volume becomes

$$R_{\rm ion} \simeq 2 \times 10^{-8} n_{\rm H} \,{\rm cm^{-3}\,s^{-1}}. \quad (3)$$

- If we assume that the temperature within the ionized region is around 10^4 K, then $\alpha \simeq 2 \times 10^{-13}$, and so we find that in equilibrium

$$2 \times 10^{-8} n_{\rm H} = 2 \times 10^{-13} n_{\rm H^+} n_{\rm e^-}. \quad (4)$$

Writing $n_{\rm H^+}/n = x$, $n_{\rm H}/n = 1 - x$ and assuming that $n_{\rm H^+} = n_{\rm e^-}$, we can reduce this to

$$(1 - x) = 10^{-5} x^2 n, \quad (5)$$

where n is the number density of hydrogen nuclei. If the number density is low, say $n = 10\,{\rm cm^{-3}}$, this is satisfied only when $x \simeq 1$; in other words, only if the neutral fraction of the gas is very small, of order 10^{-4} in this case.

- The fact that the neutral fraction is so small justifies our neglect of absorption when we are close to the star. As we move away from the star, however, this approximation becomes less accurate, as the column density of neutral hydrogen the photons must penetrate increases. Eventually, the optical depth of the ionized region to ionizing photons will approach one, leading to a significant drop in the photoionization rate and a transition from mostly ionized to mostly neutral gas.

- This transition occurs rapidly, and has a thickness corresponding to around a single mean free path of an ionizing photon. Using the same values for n and σ_ν as before, we find that this is around 0.01 pc, i.e. much, much smaller than the size of the HII region. We refer to this very thin transition region as an **ionization front**, and for many purposes we can approximate it as being infinitesimally thick (c.f. the usual treatment of hydrodynamic shock fronts).

- Let us now look at the microphysics within the HII region in a bit more detail. In the case of hydrogen, we can assume that all of the hydrogen atoms are initially in their ground state (since $n_{\rm crit}$ is very large) and can calculate the photoionization cross-section exactly. We find that at frequencies $\nu > \nu_0$,

$$\sigma_\nu = \sigma_0 \left(\frac{\nu_0}{\nu}\right)^4 \frac{\exp(4 - 4\arctan\epsilon/\epsilon)}{1 - \exp(-2\pi/\epsilon)}, \quad (6)$$

where ν_0 is the threshold energy for photoionization (i.e. $h\nu_0 = 13.6\,\mathrm{eV}$), $\sigma_0 = 6.3 \times 10^{-18}\,\mathrm{cm}^2$ is the value of σ_ν at the ionization threshold and ϵ is given by

$$\epsilon = \sqrt{\frac{\nu}{\nu_0} - 1}. \tag{7}$$

The same expression holds for any hydrogenic ion with nuclear charge Z if we make the following substitutions

$$\nu_0 \rightarrow Z^2 \nu_0, \qquad \sigma_0 \rightarrow \frac{\sigma_0}{Z^2}. \tag{8}$$

- The photoionization cross-section drops off rapidly with increasing photon energy. The complicated expression given above can be fairly well approximated at frequencies close to ν_0 by the much simpler expression

$$\sigma_\nu \simeq \sigma_0 \left(\frac{\nu_0}{\nu}\right)^3. \tag{9}$$

  We see from this that higher energy photons can penetrate significantly further into the gas than those with energies close to the ionization threshold.

- The electrons produced by photoionization have energies that depend on the energy of the photon responsible for the photoionization: $E_{\rm elec} \simeq E_\gamma - E_{\rm th}$, where $E_{\rm th}$ is the ionization potential. However, in typical HII regions, the electron-electron, electron-proton and electron-atom elastic scattering timescales are all relatively small, meaning that not only do the electron energies thermalize, but also that we can assume that the electron temperature is the same as that of the ions or the neutrals.

- When computing recombination rates, we can therefore safely assume a thermal distribution of electron velocities. The recombination coefficient for recombination to an atomic state with principal quantum number n and orbital angular momentum L can therefore be written as

$$\alpha_{nL} = \int_0^\infty v \sigma_{nL}(v) f(v)\,\mathrm{d}v, \tag{10}$$

  where σ_{nL} is the recombination cross-section and $f(v)$ is the Maxwell-Boltzmann velocity distribution

- Detailed calculations show that for hydrogen, σ_{nL} is typically of the order of 10^{-20}–$10^{-21}\,\mathrm{cm}^2$. For comparison, the physical "size" of a hydrogen atom is approximately 10^{-8} cm, leading to a geometric cross-section of order $10^{-16}\,\mathrm{cm}^2$. The recombination cross-section is therefore much smaller than the geometric cross-section.

- To understand why this is so, think about what has to happen in order to produce a bound hydrogen atom from our initially unbound proton and electron. To produce a bound state, we must be able to radiate away enough energy such that the remaining kinetic energy of the electron in the rest frame of the proton is less than the binding energy of hydrogen. To radiate away this energy, we must emit a photon, and we must do so *during* the proton-electron collision.

- We can estimate the proton-electron collision timescale as $t_{\rm coll} = L_{\rm atom}/v_{\rm e}$, where $L_{\rm atom}$ is the size of the atom and $v_{\rm e}$ is the thermal velocity of an electron. At $T = 10^4$ K, we have
$$v_{\rm e} \simeq 5 \times 10^7 \, {\rm cm \, s^{-1}}, \tag{11}$$
and hence
$$t_{\rm cross} \simeq \frac{10^{-8}}{5 \times 10^7} \simeq 2 \times 10^{-16} \, {\rm s}. \tag{12}$$

- The probability that we emit a photon during this short time period is
$$p = A_{\rm tot} t_{\rm cross}, \tag{13}$$
where $A_{\rm tot} = \sum_j A_{cj}$ is the total spontaneous radiative transition rate into all possible bound states of the atom. The latter is of order $10^9 \, {\rm s^{-1}}$, and so $p \simeq 10^{-7}$.

- We therefore see that the vast majority of electron-proton collisions do not result in photon emission and hence do not lead to recombination.

- Naively, we would expect, given the value of p that we have derived above, that the recombination cross-section should be of order $10^{-16} \times 10^{-7} = 10^{-23} \, {\rm cm^2}$. However, the true value is somewhat larger than this because of the fact that the effective collision cross-section of the electrons and protons is much larger than the geometric cross-section, owing to the Coulomb interaction between the two charged particles.

- Once we know the recombination rate coefficients for recombination into each different n, L state, we can get a total recombination rate coefficient simply by summing over all of them:
$$\alpha_{\rm A} = \sum_{n,L} \alpha_{n,L} \tag{14}$$
The rate coefficient we obtain in this way is known as the **case A** recombination rate coefficient.

- At this point, we run into the first complication: what happens to the photons produced by recombination directly into the $n = 1$ ground state? These photons have energies $h\nu > 13.6$ eV and so are capable of ionizing hydrogen.

- If the gas is optically thin to ionizing photons, or we are prepared to directly model their propagation through the ISM, then it's OK to use the case A recombination rate. However, we are often in a regime where the gas is highly optically thick to ionizing photons, meaning that any that are produced during the recombination process will be absorbed close to where the recombination took place.

- This fact forms the basis of a simple approximation known as the **on-the-spot** approximation. We assume that all of these ionizing photons are immediately reabsorbed by the gas, and hence account for them simply by modifying our calculation of the recombination rate coefficient to exclude recombinations into the $n = 1$ state, i.e.
$$\alpha_{\rm B} = \sum_{n>1,L} \alpha_{n,L} \tag{15}$$

The recombination rate coefficient we obtain in this way is known as the **case B** recombination rate coefficient.

4.1.2 Clouds containing hydrogen and helium

- In the local ISM, the abundance (by number) of helium relative to hydrogen is about 10%. Including helium therefore does not dramatically change the behaviour of the ionized gas, but does improve the accuracy of our models.
- Helium is a two electron atom and therefore has two ionization states: He^+, which has an ionization potential $h\nu_1 = 24.6$ eV, and He^{++}, which has an ionization potential of 54.4 eV.
- Hot O stars can emit significant numbers of photons with energies greater than 24.6 eV, but emit very few photons with energies greater than 54.4 eV. Therefore, we can largely neglect He^{++} in ordinary HII regions, and focus only on the balance between He and He^+.
- Including helium in our model means that there are now two sets of photons that one must keep track of: photons with $13.6 < h\nu < 24.6$ eV, which can ionize hydrogen but not helium, and photons with $h\nu > 24.6$ eV, which can ionize both hydrogen and helium.
- If the ionizing spectrum is concentrated near 13.6 eV with only a few photons above 24.6 eV, then the ionized region will consist of a small region in which both H and He are ionized, surrounded by a much larger region in which only H is ionized.
- If the ionizing spectrum is relatively flat, on the other hand, then the HII region and the HeII region largely coincide.
- Mathematically, the treatment of He charge balance is very similar to that outlined for hydrogen above. When the gas is in photoionization equilibrium, we simply balance the He photoionization rate and the recombination rate. The most important difference concerns how we treat He^+ recombination.
- As in the case of hydrogen, recombination directly to the He ground state produces photons capable of ionizing helium. However, these photons can also ionize hydrogen, and so even if they are absorbed close to their source, we cannot simply apply the "on-the-spot" approximation. Instead, we have to write the He^+ recombination coefficient as
$$\alpha_{\rm He} = y\alpha_{n=1} + \sum_{n>1,L} \alpha_{n,L} \tag{16}$$
where $\alpha_{n=1}$ is the rate coefficient for recombination directly into the He ground state, $\alpha_{n,L}$ denotes the rate coefficient for recombination into the state with principal quantum number n and orbital angular momentum L, and y is the fraction of photons produced by recombination to the $n = 1$ state that ionize hydrogen. [The remaining fraction

$(1 - y)$ ionize helium and hence have the effect of reducing $\alpha_{\rm He}$ compared to the case A rate].

- The value of y is given by

$$y = \frac{\sigma_{\rm H}(\nu_1) n_{\rm H}}{\sigma_{\rm H}(\nu_1) n_{\rm H} + \sigma_{\rm He}(\nu_1) n_{\rm He}}, \tag{17}$$

where $\sigma_{\rm H}(\nu_1)$ is the hydrogen photoionization cross-section at $h\nu_1 = 24.6$ eV, $\sigma_{\rm He}(\nu_1)$ is the helium photoionization cross-section at the same energy, $n_{\rm H}$ is the local number density of atomic hydrogen and $n_{\rm He}$ is the local number density of atomic helium.

- When the gas is mostly neutral, so that $n_{\rm He} = 0.1 n_{\rm H}$, this yields $y \simeq 0.68$ – in other words, around 70% of the photons are absorbed by hydrogen and only 30% are absorbed by helium.

- The other complication that we face is that even when the helium atom recombines to an excited state, it can still produce bound-bound photons capable of ionizing hydrogen. The details of this depend on whether the helium atom recombines into a triplet state (electron spins parallel, $S = 1$) or a singlet state (electron spins anti-parallel, $S = 0$).

- Transitions between singlet and triplet states are forbidden, and the helium ground state is a singlet state. This means that atoms that recombine into triplet states eventually wind up in the metastable $2\,^3$S triplet state. In dense gas, this state is primarily depopulated by collisional transitions to the $2\,^1$S or $2\,^1$P singlet states. At low density, on the other hand, it is primarily depopulated by forbidden single-photon transitions to the ground state, which produce photons with an energy 19.8 eV.

- Atoms that recombine into singlet states typically wind up in either the $2\,^1$S or $2\,^1$P states. Atoms in the $2\,^1$P state decay to the ground state via an allowed transition, producing a photon with energy 21.2 eV, while those in the $2\,^1$S state decay via two-photon emission, with the sum of the photon energies being 20.6 eV.

- All of these routes to the ground state can therefore produce photons capable of ionizing hydrogen. The fraction of He^+ recombinations that actually do produce photons that can ionize H depends on the importance of the $2\,^1$S – $1\,^1$S decay relative to the other routes to the ground state, and hence on the electron density. It is possible to show that in the low density limit, a fraction $p \simeq 0.96$ of all He^+ recombinations lead to hydrogen ionization, while in the high density limit this figures decreases to $p \simeq 0.66$. (For details of this calculation, see Osterbrock, "Astrophysics of gaseous nebulae and active galactic nuclei".).

- The rate at which hydrogen atoms are ionized due to the effects of He^+ recombination is therefore given by

$$R_{\rm ion} = y\alpha_{n=1} + p \sum_{n>1,L} \alpha_{n,L}. \tag{18}$$

Since $y \simeq 0.68$ and $0.66 < p < 0.96$, the rate is within a factor of two of the He^+ recombination rate. If our gas is in photodissociation equilibrium, then the latter is the same as the He^+ photoionization rate. The importance of this effect therefore depends on the ratio of the H and He photoionization rates. When the ionizing spectrum is soft, the former is much larger than the latter, and this effect is unimportant. On the other hand, if the ionizing spectrum is very hard, then the He photoionization rate can be considerable and this effect can become quite important.

4.1.3 Secondary ionization

- As we have already mentioned, the electron produced by the ionization of a hydrogen or helium atom – often referred to as a **photoelectron** – has an energy

$$E_{\rm elec} = E_\gamma - E_{\rm th}, \tag{19}$$

where E_γ is the photon energy and $E_{\rm th}$ is the ionization potential of the element in question.

- If $E_{\rm elec} > 13.6$ eV, this photoelectron has sufficient energy to collisionally ionize a hydrogen atom if it collides with one before it has a chance to thermalize. Similarly, if $E_{\rm elec} > 24.6$ eV, the photoelectron can ionize helium.

- If $E_{\rm elec} \gg 13.6$ eV, the photoelectron can potentially ionize multiple hydrogen or helium atoms before being slowed to a point at which its energy drops below 13.6 eV. In this case, these **secondary ionizations** can potentially contribute significantly to the overall photoionization rate.

- In the case of a pure hydrogen nebula, we can write the photoionization rate in a form that explicitly accounts for the effects of secondary ionizations:

$$R_{\rm ion} = n_{\rm H} \int_{\nu_0}^{\infty} \frac{4\pi J_\nu}{h\nu} \sigma_\nu({\rm H}) \left[1 + W_{\rm ion}(h\nu - h\nu_0, x)\right] {\rm d}\nu. \tag{20}$$

Here, $W_{\rm ion}$ gives the mean number of secondary ionizations per primary ionization. This is a function both of the photon energy – energetic photons produce energetic photoelectrons that can ionize more hydrogen atoms – and also the fractional ionization of the gas, x. The latter is important because one of the main way in which the photoelectrons lose energy is through electron-electron collisions. When x is large, these dominate and most of the photoelectron energy goes into heat; on the other hand, when x is small, little of the energy is lost as heat, and more secondary ionizations are produced. Note also that energy can in addition be lost through collisional excitation of bound states of hydrogen, so even in the limit $x = 0$, not all of the photoelectron energy is available for ionizing hydrogen.

- In the more realistic case of a gas with a mix of hydrogen and helium, the basic idea is similar, but the details of the energy loss are more involved. A detailed treatment of this case can be found in Dalgarno, Yan & Liu (1999, ApJS, 125, 237).

- In practice, secondary ionization is typically important only when the ionizing spectrum is very hard (e.g. in gas irradiated by X-rays). Stellar sources generally do not produce very many ionizing photons with energies sufficiently high to lead to secondary ionization, and so the contribution that this process makes to the overall photoionization rate is small.

4.1.4 Thermal balance within HII regions

- The main process responsible for heating the gas in an HII region is photoionization. As we have already mentioned, the photoelectron produced by photoionization has a kinetic energy equal to the difference between the photon energy and the ionization potential, and if the gas is highly ionized, most of this energy is transformed into heat.

- If we assume that secondary ionization is unimportant and that all of the excess energy is converted to heat, then we can write the photoionization heating rate as

$$\begin{aligned} \Gamma_{\rm ion} &= n_{\rm H} \int_{\nu_0}^{\infty} \frac{4\pi J_\nu}{h\nu} (h\nu - h\nu_0) \sigma_\nu({\rm H}) {\rm d}\nu \;\; {\rm erg\,s^{-1}\,cm^{-3}}, \quad (21) \\ &= n_{\rm H} \int_{\nu_0}^{\infty} 4\pi J_\nu \left(1 - \frac{h\nu_0}{h\nu}\right) \sigma_\nu({\rm H}) {\rm d}\nu \;\; {\rm erg\,s^{-1}\,cm^{-3}}. \quad (22) \end{aligned}$$

- In optically thin gas, illuminated with a flat spectrum, or one that decreases with increasing energy, the vast majority of photons have energies close to $h\nu_0$. In this case, the amount of energy deposited per ionization is typically of the order of an eV or less, corresponding to a gas temperature of ~ 10000 K.

- In optically thick gas, on the other hand, the photoionization, and hence the heating, is dominated by higher energy photons with frequencies close to the value at which $\tau = 1$. In this case, the heating rate per ionization can be considerably higher, resulting in gas temperatures of several times 10^4 K.

- Cooling in HII regions is dominated by the collisional excitation of the allowed transitions of hydrogen and helium, in particular the excitation of the Lyman-α line. A commonly adopted value for the cooling rate due to Lyman-α cooling is

$$\Lambda_{\rm H} = 7.5 \times 10^{-19} \exp\left(\frac{-118000}{T}\right) n_{\rm e} n_{\rm H} \, {\rm erg\,s^{-1}\,cm^{-3}}. \quad (23)$$

- The exponential term in this expression is due to the fact that an electron kinetic energy of 10.2 eV is required in order to excite the Lyman-α line. As a result, the Lyman-α cooling rate is highly temperature sensitive at $T \ll 120000$ K, and Lyman-α cooling becomes ineffective below around 8000 K.

- Atomic fine structure cooling can also operate within HII regions, but typically is not strong enough to balance photoionization heating. As a result, the temperature within most HII regions created by normal stellar sources hovers around 10000 K – cooling is too ineffective to lower it much below this value, while the optically thin nature of the ionized gas means that photoionization heating cannot raise it much above this value.

4.2 Strongren spheres

- Having looked at the microphysics of the gas within an HII region in some detail, we now step back and take a larger-scale view. We want to know how HII regions grow, and what physical processes are responsible for determining their final size.

- For simplicity, we assume in the following that our gas is composed of pure hydrogen; including the effects of helium does not materially alter the qualitative details of the models.

- We start with a very simple approach, based on a photon-counting argument. In equilibrium, the number of ionizing photons emitted per second by our central source must equal the number removed by ionizations occurring within our HII region. The latter quantity is in turn equal to the number of recombinations occurring within this region. We therefore have the expression:

$$\dot{N}_{\rm ion} = \frac{4\pi}{3} n^2 \alpha_{\rm rec} R_{\rm S}^2, \tag{24}$$

  where we have assumed a gas with constant hydrogen nuclei number density n, and where $\alpha_{\rm rec}$ is the recombination coefficient and $R_{\rm S}$ is the equilibrium radius of the HII region, known as the **Stromgren radius**.

- By rearranging Equation 24, we can obtain the following expression for $R_{\rm S}$:

$$R_{\rm S} = \left(\frac{3\dot{N}_{\rm ion}}{4\pi n^2 \alpha} \right)^{1/3}. \tag{25}$$

- In principle, recombinations can occur to all bound states of the hydrogen atom, including the $n = 1$ ground state. The recombination rate coefficient that one obtains when all of these final states are included is known as the **case A** recombination coefficient.

- In general, the mean free path of an ionizing photon within an HII region is small, and we can make use of the on-the-spot approximation. In this case, our expression for $R_{\rm S}$ becomes:

$$R_{\rm S} = \left(\frac{3\dot{N}_{\rm ion}}{4\pi n^2 \alpha_B} \right)^{1/3}, \tag{26}$$

  where $\alpha_{\rm B}$ is the case B recombination rate coefficient.

- As we have already discussed, the temperature of the ionized gas in an HII region is typically around 10^4 K. At this temperature $\alpha_B \simeq 2 \times 10^{-13} \, {\rm cm^3 \, s^{-1}}$. Suppose now we have a $20 \, {\rm M_\odot}$ O star, which produces $\dot{N}_{\rm ion} \sim 10^{48} \, {\rm s^{-1}}$ embedded in a cloud with density $n = 100 \, {\rm cm^{-3}}$. We then have $R_{\rm S} \sim 1.6 \, {\rm pc}$. There are roughly 5×10^{58} hydrogen atoms within this volume, and so it will take

$$t_{\rm ion} \simeq \frac{5 \times 10^{58}}{\dot{N}_{\rm ion}} \simeq 1600 \, {\rm yr} \tag{27}$$

  to ionize the gas.

- More massive O stars produce many more ionizing photons, and hence ionize much larger regions. However, note that since $R_S^3 \propto \dot{N}_{\rm ion}$, the ionization timescale is independent of the rate at which ionizing photons are produced, and varies only because of differences in the gas density or temperature.

- Note also that $t_{\rm ion} \ll t_{\rm MS}$, the main sequence lifetime, even for the most massive O stars – we therefore rapidly approach the Stromgren solution, unless n is very small. To make $t_{\rm ion} \sim t_{\rm MS}$, we need to assume a density $n \sim 3\,{\rm cm}^{-3}$, characteristic of the warm, diffuse ISM, and not of the dense clouds where stars form.

- What happens if our density distribution is more complicated than uniform density? The case where we have small-scale structure in the density distribution is easy to handle. Provided that the correlation length of the structures is much less than the size of our HII region, we can just account for the clumpy structure by making the replacement $n^2 \rightarrow Cn^2$, where C is the clumping factor, defined as

$$C \equiv \frac{\langle n^2 \rangle}{\langle n \rangle^2}, \tag{28}$$

where the angle brackets indicate spatial averages.

- A more interesting case occurs if we have our ionizing source located in the centre of some density gradient (which is not unreasonable, given that stars form in dense cores within molecular clouds). In this case, the behaviour depends on the steepness of the gradient.

- Suppose for simplicity that we have a spherically-symmetric density distribution, with $n = n_0 r^{-\beta}$. We can write the number of recombinations that occur per second within a sphere of radius R as

$$\dot{N}_{\rm rec} = \int_0^R 4\pi\alpha n_0^2 r^{2-2\beta}\,{\rm d}r \tag{29}$$

$$= 4\pi\alpha n_0^2 \int_0^R r^{2-2\beta}\,{\rm d}r. \tag{30}$$

- For $\beta < 1.5$, we therefore obtain

$$\dot{N}_{\rm rec} = \frac{4\pi\alpha n_0^2}{3-2\beta} R^{3-2\beta}. \tag{31}$$

- For $\beta \geq 1.5$, however, the integral diverges. In this case, the profile is so cuspy that the number of recombinations per second per unit volume tends to infinity as $r \longrightarrow 0$.

- The problem here is the singularity in our density profile, which is, after all, not physical. If we consider the more reasonable case where the density profile has a constant density core and a power-law envelope

$$n = \begin{cases} n_0 & r < r_c \\ n_0 \left(\frac{r_c}{r}\right)^\beta & r > r_c \end{cases} \tag{32}$$

then we find that

$$\dot{N}_{\rm rec} = \frac{4\pi}{3} n_0^2 \alpha r_c^3 + 4\pi n_0^2 r_c^{2\beta} \alpha \int_{r_c}^{R} r^{2-2\beta} \, {\rm d}r. \tag{33}$$

- The integral in this expression is given by

$$\frac{1}{3-2\beta} \left(R^{3-2\beta} - r_c^{3-2\beta} \right) \qquad \beta < 1.5 \tag{34}$$

$$\ln \left(\frac{R}{r_c} \right) \qquad \beta = 1.5 \tag{35}$$

$$\frac{1}{2\beta-3} \left(r_c^{3-2\beta} - R^{3-2\beta} \right) \qquad \beta > 1.5 \tag{36}$$

- For $\beta \leq 1.5$, we see that $\dot{N}_{\rm rec} \rightarrow \infty$ and $R \rightarrow \infty$, and hence can conclude that regardless of the rate at which our source produces ionizing photons, there must be *some* radius at which we can balance this with recombinations; in other words, there is always a well-defined Stromgren radius.

- For $\beta > 1.5$, however, this is not the case. If $\beta > 1.5$ and $R \rightarrow \infty$, the number of recombinations per second tends towards a finite value

$$\dot{N}_{\rm rec} \rightarrow 4\pi n_0^2 \alpha r_c^3 \left(\frac{1}{3} + \frac{1}{2\beta - 3} \right) \equiv \dot{N}_{\rm rec,crit}. \tag{37}$$

  If $\dot{N}_{\rm ion} > \dot{N}_{\rm rec,crit}$, then there is no solution for $R_{\rm S}$ – recombinations can never balance ionizations, and so the whole of the density distribution will eventually become ionized.

5 Dynamical evolution of HII regions

- So far, we've been ignoring the dynamics of the gas and assuming it remains at rest. However, if we have a constant or radially decreasing density distribution, it is plain that this cannot be the case – the ionized gas is hotter than the neutral gas and so it must also be over-pressured with respect to the neutral gas.

- The pressure gradient created by the difference in temperatures acts to accelerate the gas outwards, away from the ionizing source. The maximum velocity that the ionized gas will obtain is comparable to the speed of sound in the ionized gas, i.e. around $10 \, {\rm km \, s^{-1}}$. If the expansion speed of the ionization front is much larger than this, then it is a good approximation to ignore the gas dynamics.

- What is the expansion speed of the I-front? In the Stromgren analysis, the number of photons available to cause new ionizations (rather than simply counteracting the effect of recombinations) in a time Δt is given by

$$N_{\rm ion} = \left(\dot{N}_{\rm ion} - \frac{4\pi}{3} n^2 \alpha R_I^3 \right) \Delta t, \tag{38}$$

  where we are once again considering our constant-density HII region; the analysis can be extended to power-law density profiles without too much difficulty.

- This number of photons will ionize a shell with thickness ΔR_I and volume $4\pi R_I^2 \Delta R_I$ (where we assume that $\Delta R_I \ll R_I$; we can always take Δt small enough that this is certain to be the case). The number of atoms in this shell is just $4\pi R_I^2 \Delta R_I n$, and this must equal the number of photons available to cause ionizations. We therefore have:

$$4\pi R_I^2 \Delta R_I n = \left(\dot{N}_{\rm ion} - \frac{4\pi}{3} n^2 \alpha R_I^3 \right) \Delta t. \tag{39}$$

- Rearranging this expression, and taking the limit $\Delta t \rightarrow 0$, we obtain the following expression for the expansion velocity of our HII region:

$$\frac{{\rm d}R_I}{{\rm d}t} = \frac{1}{4\pi R_I^2 n} \left(\dot{N}_{\rm ion} - \frac{4\pi}{3} n^2 \alpha R_I^3 \right). \tag{40}$$

- When $R_I \ll R_S$, we know that the first term in the parentheses dominates, and we simply have

$$\frac{{\rm d}R_I}{{\rm d}t} = \frac{\dot{N}_{\rm ion}}{4\pi R_I^2 n}. \tag{41}$$

- Using the same example figures as before – $\dot{N}_{\rm ion} = 10^{48}\ {\rm s}^{-1}$, $n = 100\ {\rm cm}^{-3}$ – we find that

$$\frac{{\rm d}R_I}{{\rm d}t} \simeq 1000 \left(\frac{R_I}{1\ {\rm pc}} \right)^{-2}\ {\rm km\, s}^{-1}. \tag{42}$$

  Hence, if $R_I \ll R_S$, and R_I is also small, the expansion speed of the HII region can be orders of magnitude faster than the sound speed of the ionized gas. In this case, it is often a good approximation to ignore the outward expansion of the gas.

- However, we know that the speed at which the ionization front expands must become small once $R \simeq R_{\rm S}$ (since it is zero when $R = R_{\rm S}$, at least in the Stromgren analysis). Therefore, provided that we are in a regime where $t_{\rm ion} \ll t_{\rm MS}$, it is clear that the effects of the gas dynamics will eventually become important.

- To treat the dynamics of the gas plus the ionization front, we start by writing down the continuity equation for the electrons:

$$\frac{\partial n_{\rm e}}{\partial t} + \vec{\nabla} \cdot (n_{\rm e} \vec{u}) = I - R. \tag{43}$$

  Here $\vec{u}$ is the velocity along a radial ray pointing outwards from the source, I is the ionization rate per unit volume, R is the recombination rate per unit volume, and we have assumed that there are no significant motions perpendicular to our radial ray.

- For the recombination rate R, we have $R = \alpha n_{\rm e} n_{\rm H^+}$, but if our gas is composed purely of hydrogen, then $n_{\rm e} = n_{\rm H^+}$ and we can write R in a simpler form as $R = \alpha n_e^2$.

- To determine I, we make several assumptions. We assume that the only important source of ionizing radiation within the volume of interest is our central ionizing source (and hence ignore the effect of any diffuse field generated within the HII region), that the propagation speed of our photons is infinite and that the I-front absorbs all photons that reach it.

- Let $V_{\rm if}$ be the volume associated with our ionization front. Then the number of ionizations occurring within this volume per unit time is given by $IV_{\rm if}$. However, according to our assumptions, this must equal the number of ionizing photons *reaching* the front per unit time, i.e.

$$IV_{\rm if} = -\int_S \vec{J} \cdot \vec{n} dS, \tag{44}$$

where S is the surface area of the front, n is a unit vector perpendicular to the front, and where $\vec{J}$ is the flux of ionizing photons, given by

$$\vec{J} = \frac{\dot{N}_{\rm ion} e^{-\tau}}{4\pi r^2} \vec{e_r}, \tag{45}$$

where r is the distance to the source, and $\vec{e_r}$ is the radial unit vector.

- Applying Gauss' theorem allows us to rewrite Equation 44 as

$$IV_{\rm if} = -\int_V \vec{\nabla} \cdot \vec{J} dV. \tag{46}$$

We next note that instead of considering the whole of the ionization front, we could just apply the same argument to a very small patch of it. Provided we make the patch sufficiently small, we can ignore any variation of $\vec{J}$ within it, allowing us to write

$$IV = -(\vec{\nabla} \cdot \vec{J})V, \tag{47}$$

from which the relation

$$I = -\vec{\nabla} \cdot \vec{J} \tag{48}$$

follows trivially.

- We can therefore write our electron continuity equation as

$$\frac{\partial n_{\rm e}}{\partial t} + \vec{\nabla} \cdot (n_{\rm e} \vec{u}) + \vec{\nabla} \cdot \vec{J} + \alpha n_{\rm e}^2 = 0. \tag{49}$$

- Let us suppose that at some moment in time, the ionization front is moving with a speed U. If we consider a small cylindrical volume V centred on the front and moving with the front velocity, then integration of Equation 49 within this volume yields:

$$\int_V \left[\frac{\partial n_{\rm e}}{\partial t} + \vec{\nabla} \cdot n_{\rm e}(\vec{u} - \vec{U}) + \vec{\nabla} \cdot \vec{J} + \alpha n_{\rm e}^2 \right] {\rm d}V = 0. \tag{50}$$

If we consider the steady-state solution, and take our volume to be infinitesimally small, then the first and fourth terms vanish and we have:

$$\int_V \left[\vec{\nabla} \cdot n_{\rm e}(\vec{u} - \vec{U}) + \vec{\nabla} \cdot \vec{J} \right] {\rm d}V = 0. \tag{51}$$

- We can next apply Gauss' theorem to turn this into a surface integral. If we take the sides of our volume in the direction perpendicular to the front to be infinitesimally short, then we obtain:
$$\int_A \left[n_{\rm e}(\vec{u} - \vec{U}) + \vec{J} \right] \cdot {\rm d}\vec{A} = 0, \tag{52}$$
where $\vec{A}$ is the area parallel to the front. From this, the ionization front jump condition directly follows:
$$n_{e,2}(u_2 - U) + J_2 = n_{e,1}(u_1 - U) + J_1 \tag{53}$$
where subscript 1 denotes pre-front conditions (i.e. outside of the HII region) and subscript 2 denotes post-front conditions (i.e. within the HII region).

- For simplicity, we assume that the gas ahead of the ionization front is completely neutral and that no ionizing photons penetrate into this region. In this case, $n_{e,1} = 0$ and $J_1 = 0$, and we have:
$$n_{e,2}(u_2 - U) + J_2 = 0. \tag{54}$$

- The number of hydrogen atoms per second that flow into the front is given by
$$\frac{\rho_1(u_1 - U)}{m_{\rm H}}. \tag{55}$$
This must equal the number of electrons flowing away, $n_{e,2}(u_2 - U)$, and so we have:
$$J_2 = -\frac{\rho_1(u_1 - U)}{m_{\rm H}}, \tag{56}$$
$$= \frac{\rho_1 v_1}{m_{\rm H}}, \tag{57}$$
where $v_1 = U - u_1$ is the speed at which gas is flowing into the front, i.e. the relative speed at which the front is moving through the undisturbed gas.

- Clearly, when J_1 is large or ρ_1 is small, the speed of the front can be very large, as we saw in our simple example above.

- In addition to our ionization front jump condition, we also have jump conditions that are implied by the fact that mass and momentum are conserved across the front (provided we can ignore the momentum imparted to the gas by the radiation pressure):
$$\rho_2 v_2 = \rho_1 v_1, \tag{58}$$
$$p_2 + \rho_2 v_2^2 = p_1 + \rho_1 v_1^2. \tag{59}$$
These conditions should be familiar – they are two of the Rankine-Hugoniot conditions that relate pre-shock and post-shock fluid quantities. However, they do not hold only for shocks – they are a consequence of the conservation laws, and hold across any arbitrary interface.

- The third of the Rankine-Hugoniot conditions, which equates the pre-shock and post-shock energies, does not hold across an ionization front. The energy of the gas is **not conserved** across the front, as it gains energy from the radiation field. (NB. The total energy of gas plus radiation field *is* still conserved, of course).

- In its place, we need to make some assumption about the thermal conditions in the pre-shock and post-shock gas. Here, for simplicity, we assume that the pre-shock gas is isothermal, with sound speed $c_{s,1}$ and that the post-shock gas is also isothermal (albeit at a different temperature), with sound speed $c_{s,2}$.

- This allows us to write the pre- and post-front pressures as:

$$p_1 = \rho_1 c_{s,1}^2, \qquad\qquad p_2 = \rho_2 c_{s,2}^2. \tag{60}$$

  The second of the Rankine-Hugoniot conditions therefore becomes

$$\rho_2 \left(c_{s,2}^2 + v_2^2\right) = \rho_1 \left(c_{s,1}^2 + v_1^2\right). \tag{61}$$

- Some algebra (and use of the first RH condition) allows us to write this as

$$\frac{\rho_2}{\rho_1} = \frac{1}{2c_{s,2}^2}\left[\left(c_{s,1}^2 + v_1^2\right) \pm \left\{\left(c_{s,1}^2 + v_1^2\right)^2 - 4c_{s,2}^2 v_1^2\right\}^{1/2}\right] = \frac{v_1}{v_2}. \tag{62}$$

- For this to yield a real value for the density contrast (i.e. a physical solution), the term in the square-root must be positive (or zero). Hence:

$$\begin{aligned} \left(c_{s,1}^2 + v_1^2\right)^2 &\geq 4c_{s,2}^2 v_1^2, & (63)\\ c_{s,1}^2 + v_1^2 &\geq 2c_{s,2} v_1. & (64) \end{aligned}$$

- If v_1 is large, this reduces to the constraint

$$v_1 \geq 2c_{s,2}, \tag{65}$$

  and hence a solution exists for large v_1. A solution also exists in the limit of small v_1: as $v_1 \rightarrow 0$, the right-hand side also tends to zero, while the left-hand side tends to $c_{s,1}^2$. However, there are intermediate values of v_1 for which the argument of the square-root is negative, and no physical solution exists.

- The critical values of v_1 for which the square-root term is zero are given by

$$v_{\pm}^2 = \left[c_{s,2} \pm \left(c_{s,2}^2 - c_{s,1}^2\right)^{1/2}\right]^2. \tag{66}$$

- We call the larger root of this equation v_R and the smaller root v_D:

$$\begin{aligned} v_R = &\ c_{s,2} + \left(c_{s,2}^2 - c_{s,1}^2\right)^{1/2} \simeq 2c_{s,2} & (67)\\ v_D = &\ c_{s,2} - \left(c_{s,2}^2 - c_{s,1}^2\right)^{1/2} \simeq \frac{c_{s,1}^2}{2c_{s,2}}, & (68) \end{aligned}$$

  where the approximate equalities follow if we assume that $c_{s,2} \gg c_{s,1}$.

- Ionization front solutions with $v_1 \geq v_R$ are known as **R-type** fronts. Those with $v_1 \leq v_D$ are known as **D-type** fronts.

- In terms of v_R, v_D, our expression for the density contrast becomes

$$\frac{\rho_2}{\rho_1} = \frac{1}{2c_{s,2}^2}\left[\left(v_R v_D + v_1^2\right) \pm \left\{\left(v_1^2 - v_R^2\right)\left(v_1^2 - v_D^2\right)\right\}^{1/2}\right]. \tag{69}$$

- Solutions for which we take the positive sign in this equation are **strong**; solutions for which we take the negative sign as **weak**.

- To solve for the density contrast, we need to know the two sound speeds and also v_1. The latter is set by J_1, the ionizing photon flux.

- During the initial expansion phase, we have already seen that v_1 is very large; we easily satisfy $v_1 \gg c_{s,2}$ and hence start as an R-type front. If gas density initially uniform, then gas won't have time to re-adjust structure in response to pressure gradients. Hence we have a **weak R-type** front.

- As front expands, speed drops, until $v_1 = v_R$; at this point, we have what is known as an **R-critical** front. Now what? The speed can't drop further, as we'd enter the unphysical regime. However, the gas within the HII region is expanding with a velocity $\sim c_{s,2}$, and a sound wave is propagating ahead of the expanding region with a relative velocity that is again $\sim c_{s,2}$. What happens when we become R-critical is that the pressure wave starts to move as fast as the I-front.

- Once the pressure wave overtakes the I-front, it moves into a region where the sound speed is only $c_{s,1}$. Its motion is supersonic in this region, and so it steepens into a shock. We therefore have a three-part structure: if we picture the gas as flowing from left to right, then from right to left, we have (i) undisturbed neutral gas, (ii) shocked neutral gas, and (iii) ionized gas.

- The layer of compressed HI between the shock and the ionization front remains relatively thin during the evolution, as the speed difference between the shock and the front is not large.

- The evolution of the D-type front during this second phase can be approximated analytically with the formula

$$R_I = \left[1 + \frac{7}{4}\frac{c_{s,2}}{R_{\rm S,init}}(t - t_0)\right]^{4/7} R_{\rm S,init} \tag{70}$$

  where $R_{\rm S,init}$ is the initial Stromgren radius (i.e. the radius we obtain from our Stromgren analysis), and t_0 is the time at which the transition from R-type to D-type evolution occurs.

- In principle, expansion of the HII region will continue until the pressure of the ionized gas equals the pressure of the cold atomic gas. The density of the gas at this time is given by the condition
$$2n_{\rm final}T_2 = n_{\rm init}T_1, \tag{71}$$
where T_1 is the temperature of the neutral gas and T_2 is the temperature of the ionized gas. As the size of the Stromgren radius scales with density as $R_S \propto n^{-2/3}$, this implies that the HII region must expand by a factor
$$f_{\rm exp} = \left(\frac{2T_2}{T_1}\right)^{2/3}. \tag{72}$$
Taking 100 K for the former and 10^4 K for the latter, we find that $f_{\rm exp} \sim 34$.

- In practice, we only reach this final state if our initial Stromgren radius was small (e.g. if we're in a high density environment). For $R_{\rm S,init} = 1$ pc and $c_{s,2} = 10\ {\rm km\,s^{-1}}$, Equation 70 yields
$$\frac{R_I}{R_{\rm S,init}} \sim 11\left(\frac{t}{1\ {\rm Myr}}\right)^{4/7}, \tag{73}$$
where we have assumed that $t_0 \ll t$. For a massive O-star, this does not reach 34 within the lifetime of the star.