

# 1 Radiative processes

## 1.1 Atomic structure, selection rules

- We begin with a brief reminder of the basics of atomic structure, and the rules governing allowed and forbidden transitions. This material should hopefully be familiar from previous quantum mechanics courses. For those of you who are interested in a more detailed overview, there is a set of notes available on the course website, and I can also recommend the textbook by Atkins & Friedman<sup>1</sup>.

### 1.1.1 Atomic structure

- Atomic lines are produced by transitions between different atomic states. These states can be characterised by a set of **quantum numbers**. The most important of these quantum numbers are:

$n$  – the principal quantum number

$l$  – the orbital angular momentum quantum number

$s$  – the electron spin angular momentum

$j$  – the total angular momentum

- $n$  has values  $1, 2, \dots$  and describes which of the possible radial eigenfunctions describes the atomic state. It has the largest influence on the energy of the state.
- $l$  has values  $0, 1, \dots, n-1$  and describes the orbital angular momentum of the state. For a multi-electron atom, there is also a quantum number  $L$  describing the total orbital angular momentum, which is given by the **vector sum** of the individual orbital angular momenta.
- $s$  has values  $-1/2, +1/2$  for a single electron. For multi-electron atoms, there is also an associated quantum number  $S$  describing the total spin, given by the vector sum of the individual spins.
- $j$  is given by the vector sum of  $l$  and  $s$ . Similarly,  $J$ , the total angular momentum of a multi-electron atom, is given by the vector sum of  $L$  and  $S$ .
- Although the energies of different atomic states are typically determined primarily by  $n$ , they also depend on  $J$  because of an effect known as **spin-orbit coupling**. This coupling occurs because the electron has a magnetic moment due to its orbital motion and also an intrinsic magnetic momentum due to its spin. States where these magnetic moments are parallel have higher energy than states where they are anti-parallel.
- The resulting splitting of energy levels that would otherwise be degenerate is known as atomic **fine structure**.

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<sup>1</sup>Atkins & Friedman, *Molecular Quantum Mechanics* (5th Edition), Oxford University Press

- Note that not all states have fine structure: we need both non-zero  $L$  and also non-zero  $S$ , or the spin-orbit coupling term in the Hamiltonian vanishes. For example: the ground state of the hydrogen atom has  $S = 1/2$  but  $L = 0$ , and hence has no fine structure. On the other hand, the ground state of neutral atomic carbon has  $L = 1$  and  $S = 1$  and hence does have fine structure.
- Atoms may also have **hyperfine structure**, caused by **spin-spin coupling** (i.e. interaction between the electron spin and the nuclear spin). In general, this is of little astrophysical importance, with one extremely large exception: atomic hydrogen.
- In atomic hydrogen, the hyperfine splitting of the ground-state results in two closely separated states with an energy difference of  $5.87 \times 10^{-6}$  eV, corresponding to a wavelength of approximately 21 cm. The spectral line produced by transitions between the two hyperfine states is our main observational tool for studying neutral hydrogen, both in our Milky Way and beyond.

### 1.1.2 Selection rules for atomic transitions

- The probability for a transition to occur between two different atomic states is directly proportional to the size of the transition matrix element linking these two states. This can be derived using the techniques of time-dependent perturbation theory, but doing so is outside the scope of the current lecture course<sup>2</sup>. Here, we simply state a few important results, without proof.
- The most probable transitions are those known as **electric dipole** transitions (often simply referred to as dipole transitions). These are transitions for which the transition dipole moment is non-zero. For a transition between an initial state  $|i\rangle$  and a final state  $|f\rangle$ , the transition dipole moment is defined as

$$\vec{\mu}_{if} = \langle i | \vec{\mu} | f \rangle, \quad (1)$$

where  $\vec{\mu} = -e\vec{r}$  is the electric dipole operator.

- Transitions for which  $\vec{\mu}_{if} \neq 0$  are known as **allowed** transitions, while those for which  $\vec{\mu}_{if} = 0$  are known as **forbidden** transitions.
- Symmetry considerations allow us to determine the transitions for which the transition dipole moment is non-zero (i.e. for which there is an allowed transition). The results are easily summarized as a set of **selection rules**. The most important ones are:
  - (i)  $\Delta J = 0, \pm 1$  (but transitions with  $J = 0 \rightarrow J = 0$  are forbidden)
  - (ii) Parity change
  - (iii) One electron jumping,  $\Delta l = \pm 1$ ,  $\Delta n$  arbitrary
  - (iv)  $\Delta S = 0$

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<sup>2</sup>For an in-depth discussion, see Atkins & Friedman, Chapter 6.

(v)  $\Delta L = 0, \pm 1$  (but transitions with  $L = 0 \rightarrow L = 0$  are forbidden)

- The classic example of an allowed transition is the Lyman- $\alpha$  transition in hydrogen. The electron jumps from a state with  $l = 1$  and  $n = 2$  to  $l = 0$  and  $n = 1$ . The system changes parity during this process, and we also see that  $\Delta l = -1$ ,  $\Delta L = -1$  and  $\Delta n = 1$ . The spin quantum number  $S$  does not change, so the spin selection rule is satisfied, and hence the total angular momentum selection rule is also satisfied.
- On the other hand, if we apply the selection rules, we find that transitions between fine structure levels with the same  $n$  are forbidden, as they do not involve a change in  $l$ .
- Transitions which are forbidden at the electric dipole level may still occur, if the magnetic dipole transition matrix element

$$\vec{\mu}_{if} = \langle i | \vec{L} | f \rangle, \quad (2)$$

is non-zero, where  $\vec{L}$  is the total orbital angular momentum operator.

- Magnetic dipole transitions have their own set of selection rules:
  - (i)  $\Delta J = 0, \pm 1$  (but transitions with  $J = 0 \rightarrow J = 0$  are forbidden)
  - (ii) No parity change
  - (iii) For all electrons,  $\Delta l = 0$ ,  $\Delta n = 0$
  - (iv)  $\Delta S = 0$
  - (v)  $\Delta L = 0$ ,  $\Delta J = \pm 1$
- We see that fine structure transitions, which involves changes with  $\Delta n = 0$  and  $\Delta l = 0$ , are allowed under these selection rules. Fine structure transitions are therefore generally magnetic dipole transitions.
- An important point to remember is that magnetic dipole transitions are much weaker than electric dipole transitions (i.e. they have much smaller transition matrix elements, and hence much smaller probabilities of occurring). Consequently, forbidden transitions are generally much weaker than allowed transitions.
- Another possible type of forbidden transition is an electric dipole transition. These are weaker still than the magnetic dipole transitions, and obey yet another set of selection rules (see e.g. Tielens, *Physics and Chemistry of the Interstellar Medium*, Table 2.2). However, these transitions are of limited importance for understanding the ISM, so we will not discuss them further here.

## 1.2 Einstein coefficients

- Having refreshed our memory about the basics of atomic structure, our next step is to discuss in detail the physics of the absorption and emission of radiation from atoms (and, later, from molecules).

- We start by defining a quantity known as the **specific intensity** (or sometimes as the **radiative intensity**) of a radiation field. The specific intensity in a direction  $\vec{n}$  is the radiative energy flowing in that direction per unit time, per unit area and per unit solid angle. It is written as  $I(\vec{n})$  and has units of  $\text{W m}^{-2} \text{sr}^{-1}$  (SI) or  $\text{erg s}^{-1} \text{cm}^{-2} \text{sr}^{-1}$  (cgs).
- When considering the emission or absorption of radiation by atoms or molecules, it is generally more convenient to work in terms of the monochromatic version of the specific intensity,  $I_\nu(\nu; \vec{n})$ . This is defined such that  $I_\nu d\nu$  is the specific intensity of photons with frequencies in the range  $\nu$  to  $\nu + d\nu$ . Hence,

$$I(\vec{n}) = \int_0^\infty I_\nu(\nu; \vec{n}) d\nu. \quad (3)$$

The monochromatic version of the specific intensity has units  $\text{W m}^{-2} \text{Hz}^{-1} \text{sr}^{-1}$  (SI) or  $\text{erg s}^{-1} \text{cm}^{-2} \text{Hz}^{-1} \text{sr}^{-1}$  (cgs).

- Let us now consider two quantum states of an atom, an initial state  $|i\rangle$  and a final state  $|f\rangle$ , where  $E_i < E_f$ . The transition rate<sup>3</sup> between these two states (i.e. the rate of change of the probability of finding the atom in the final state) can be written as

$$W_{if} = B_{if} I_\nu(\nu_{if}), \quad (5)$$

where  $h\nu_{if} = E_f - E_i$ , and where  $B_{if}$  is a quantity known as the **Einstein coefficient of stimulated absorption**.

- Radiation can also cause a transition in the opposite direction, from  $|f\rangle$  to  $|i\rangle$ , resulting in the emission of a photon. This process is known as **stimulated emission**, and the rate at which it occurs can be written as

$$W_{fi} = B_{fi} I_\nu(\nu_{if}), \quad (6)$$

where  $B_{fi}$  is the **Einstein coefficient of stimulated emission**.

- If  $|i\rangle$  and  $|f\rangle$  are non-degenerate – i.e. if there is only one energy level corresponding to each state – then  $B_{if} = B_{fi}$ . In the more general case in which one or both states are degenerate, the relationship is instead given by

$$g_i B_{if} = g_f B_{fi}, \quad (7)$$

where  $g_i$  is the statistical weight of state  $|i\rangle$  and  $g_f$  is the statistical weight of state  $|f\rangle$ .

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<sup>3</sup>Note: the reason that it is useful to work in terms of the transition rate is that when we're dealing with a large number of atoms – which is often the case in astrophysics – the rate per unit volume at which the transition occurs is simply

$$R_{if} = W_{if} n_i, \quad (4)$$

where  $n_i$  is the number density of atoms in state  $|i\rangle$ . In other words, provided we're averaging over a large enough ensemble, it allows us to forget that we're dealing with the probabilities of discrete events and treat the transition rate as a continuous quantity.

- Now suppose we have an ensemble consisting of a large number of atoms, with  $N_i$  in state  $|i\rangle$  and  $N_f$  in state  $|f\rangle$ . Let us further assume that the atoms in this ensemble are in thermal equilibrium both with each other and with the radiation field. We know from statistical mechanics that the ratio of  $N_i$  and  $N_f$  in this case is given by the Boltzmann distribution:

$$\frac{N_f}{N_i} = \frac{g_f}{g_i} e^{-E_{if}/kT}, \quad (8)$$

where  $E_{if} = E_f - E_i$ ,  $k$  is Boltzmann's constant and  $T$  is the temperature of the system.

- If stimulated emission and stimulated absorption were the only processes operating to change the state of each atom from  $|i\rangle$  to  $|f\rangle$  or vice versa, then in thermal equilibrium, the two rates would be equal, i.e.

$$N_i W_{if} = N_f W_{fi}. \quad (9)$$

This implies that

$$N_i B_{if} I_\nu(\nu_{if}) = N_f B_{fi} I_\nu(\nu_{fi}), \quad (10)$$

and hence that

$$\frac{N_f}{N_i} = \frac{B_{if}}{B_{fi}}. \quad (11)$$

However, from Equation 7, we see that this would imply that

$$\frac{N_f}{N_i} = \frac{g_f}{g_i}, \quad (12)$$

which is incompatible with Equation 8! Clearly, something has gone wrong with our argument.

- Einstein resolved this conflict by postulating that in addition to stimulated emission and stimulated absorption, there was a third process acting, which he termed **spontaneous emission**.<sup>4</sup> This is a radiative transition from  $|f\rangle$  to  $|i\rangle$  that is *independent* of the strength of the radiation field. It is quantified by  $A_{fi}$ , the **Einstein coefficient of spontaneous emission**.
- The transition rate from  $|f\rangle$  to  $|i\rangle$  due to spontaneous emission is given by

$$W_{fi}^{\text{spon}} = A_{fi}, \quad (13)$$

and so in thermal equilibrium

$$N_i W_{if} = N_f (W_{fi} + W_{fi}^{\text{spon}}). \quad (14)$$

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<sup>4</sup>Note that the conservation of energy does not allow spontaneous absorption.

Therefore,

$$\frac{N_f}{N_i} = \frac{W_{if}}{W_{fi} + W_{fi}^{\text{spon}}}, \quad (15)$$

$$= \frac{B_{if}I_\nu(\nu_{if})}{B_{fi}I_\nu(\nu_{if}) + A_{fi}}, \quad (16)$$

$$= \frac{(g_f/g_i)I_\nu(\nu_{if})}{I_\nu(\nu_{if}) + A_{fi}/B_{fi}}, \quad (17)$$

$$= \frac{g_f}{g_i} \frac{1}{1 + A_{fi}/(B_{fi}I_\nu(\nu_{if}))}. \quad (18)$$

This is consistent with the Boltzmann distribution, provided that

$$A_{fi} = I_\nu \left[ \exp\left(\frac{E_{if}}{kT}\right) - 1 \right] B_{fi}. \quad (19)$$

In thermal equilibrium, the specific intensity of the radiation field is given by the Planck function:  $I_\nu = B_\nu(T)$ , where

$$B_\nu(T) = \frac{2h\nu_{if}^3/c^2}{\exp\left(\frac{E_{if}}{kT}\right) - 1}. \quad (20)$$

Hence, we find that

$$A_{fi} = \frac{2h\nu_{if}^3}{c^2} B_{fi}. \quad (21)$$

- **Important note:** our discussion of the Einstein coefficients above uses the convention that the  $B_{if}$  and  $B_{fi}$  coefficients are the constants of proportionality relating the **specific intensity** of the radiation field to the transition rates for absorption and for stimulated emission. However, some books and papers use a different convention, where they relate the transition rates to the **energy density** of the radiation field. In this other convention, we have

$$W_{if} = B_{if}u_\nu(\nu_{if}), \quad (22)$$

$$W_{fi} = A_{fi} + B_{fi}u_\nu(\nu_{if}), \quad (23)$$

where

$$u_\nu(\nu_{if}) = \frac{4\pi}{c} I_\nu(\nu_{if}). \quad (24)$$

Therefore, when using this convention, our values for  $B_{if}$  and  $B_{fi}$  are a factor of  $c/4\pi$  larger than when we use the convention based on specific intensities. Consequently, the relationship between  $A_{fi}$  and  $B_{fi}$  is different (since the value of  $A_{fi}$  can't depend on whether we use  $I_\nu$  or  $u_\nu$  to quantify the radiation field), and is given by

$$A_{fi} = \frac{8\pi h\nu_{if}^3}{c^2} B_{fi}. \quad (25)$$

- Further points to note regarding the Einstein coefficients:
  - If we have two states  $|u\rangle$  and  $|l\rangle$ , with  $E_u > E_l$ , then the characteristic lifetime of state  $|u\rangle$  is given approximately by  $t \sim A_{ul}^{-1}$ . If there are multiple lower energy states that state  $|u\rangle$  can reach via radiative decay, then the lifetime is given approximately by  $t \sim (\sum_l A_{ul})^{-1}$ , where we sum over all accessible lower-energy states.
  - The importance of absorption and stimulated emission depend on the strength of the radiation field. In particular, if the strength of the field at the frequency of the transition is much smaller than the Planck function at that frequency, then the effects of stimulated emission are generally negligible.
  - The spontaneous transition rate depends on the transition dipole moment (through  $B_{fi}$ ), but also on  $\nu^3$ . High frequency transitions – those between widely separated energy levels – therefore occur much more rapidly than low frequency transitions.
  - Fine structure transitions are therefore slow for two reasons – they are forbidden, and hence have small probabilities of occurring, but they are also low frequency transitions.
  - As an example, let us compare the dipole  $2p \rightarrow 1s$  transition in hydrogen (the Lyman- $\alpha$  line) with the ground-state fine structure transition in  $C^+$ . The Lyman- $\alpha$  transition is a permitted transition with an energy of 10.2 eV, and an Einstein coefficient  $A_{21} = 6.3 \times 10^8 \text{ s}^{-1}$ . On the other hand, the  $C^+$  fine-structure transition is a magnetic dipole transition, between two states separated by roughly 0.008 eV. Magnetic dipole transitions are typically a factor of  $\alpha^2 \sim 5 \times 10^{-5}$  less probable than electronic dipole transitions, and hence we expect the transition rate for the  $C^+$  line to be

$$A_{\text{fs}} \sim 5 \times 10^{-5} \times \left(\frac{0.008}{10.2}\right)^3 \times 6.3 \times 10^8 \text{ s}^{-1} \quad (26)$$

$$\sim 5 \times 10^{-5} \times 5 \times 10^{-10} \times 6.3 \times 10^8 \text{ s}^{-1} \quad (27)$$

$$\sim 10^{-5} \text{ s}^{-1}. \quad (28)$$

For reference, the actual value is  $2.3 \times 10^{-6} \text{ s}^{-1}$ .

- Similar considerations indicate that the spontaneous transition rate for the 21 cm line of hydrogen must be very small, since the energy difference is so small. In fact, the rate is  $\sim 3 \times 10^{-15} \text{ s}^{-1}$ .

## 1.3 Radiative transfer basics

### 1.3.1 Absorption and emission coefficients

- Another way in which to describe the interaction of radiation with matter is in terms of an absorption coefficient.

- Consider a cylindrical region with length  $ds$ , cross-sectional area  $dA$  and volume  $dV = ds \times dA$ . Assume that this region contains identical atoms with number density  $n$ , and that these atoms have an absorption cross-section at frequency  $\nu$  given by  $\sigma_\nu$ .
- Now illuminate the cylinder with a radiation field with specific intensity  $I_\nu$ . Provided that our volume is sufficiently small, we can take the value of  $I_\nu$  to be the same at both ends, and can hence write down the rate at which energy in the frequency range  $\nu \rightarrow \nu + d\nu$  and the direction  $\Omega$  absorbed by a single atom:

$$de_\nu = I_\nu(\Omega)\sigma_\nu(\Omega)d\nu d\Omega. \quad (29)$$

The total amount of energy removed in the direction  $\Omega$  is thus given by

$$dE_\nu = [nI_\nu(\Omega)\sigma_\nu(\Omega)ds] dA dt d\nu d\Omega. \quad (30)$$

We can also write this as

$$dE_\nu = \alpha_\nu(\Omega)I_\nu(\Omega)dV dt d\nu d\Omega. \quad (31)$$

where

$$\alpha_\nu(\Omega) = n\sigma_\nu(\Omega) \quad (32)$$

is a quantity known as the **absorption coefficient**.

- The quantity in square brackets in Equation 30 is simply the change in  $I_\nu$  from one end of the cylinder to the other, which can be written as

$$dI_\nu = \alpha_\nu(\Omega)I_\nu(\Omega)ds. \quad (33)$$

- In the absence of emission, we can therefore write down the following transfer equation for  $I_\nu$ :

$$\frac{dI_\nu}{ds} = -\alpha_\nu I_\nu. \quad (34)$$

If we introduce the optical depth element

$$d\tau_\nu = \alpha_\nu ds, \quad (35)$$

and the total optical depth

$$\tau_\nu = \int_0^L \alpha_\nu(s)ds, \quad (36)$$

then we can easily write down the formal solution for the attenuation of radiation within an absorbing medium in the absence of emission:

$$I_\nu(\tau_\nu) = I_{\nu,0}e^{-\tau_\nu}. \quad (37)$$

Here,  $I_{\nu,0}$  is the intensity of the radiation field incident on the medium (i.e. our  $\tau = 0$  boundary condition).



- It is sometimes convenient to work in terms of the **opacity**,  $\kappa_\nu$ , which is defined in terms of the absorption coefficient as

$$\kappa_\nu = \frac{\alpha_\nu}{\rho}. \quad (38)$$

This quantity has the advantage that it does not directly depend on the density of the medium. In terms of the opacity, the optical depth becomes:

$$\tau_\nu = \int_0^\infty \rho \kappa_\nu ds. \quad (39)$$

In the simple case in which  $\kappa_\nu$  is independent of position, this becomes

$$\tau_\nu = \kappa_\nu \int_0^\infty \rho ds = \kappa_\nu \Sigma, \quad (40)$$

where  $\Sigma$  is the column density of the absorbing medium.

- Our next step is to determine how to express  $\sigma_\nu$  and  $\alpha_\nu$  as functions of our Einstein coefficients. We know that for a single atom, the rate at which photons are absorbed in some transition  $l \rightarrow u$  can be written as

$$W_{lu} = B_{lu} J_\nu(\nu_{lu}), \quad (41)$$

where  $\nu_{lu}$  is the frequency corresponding to this transition, and where  $J_\nu$  is the mean specific intensity,

$$J_\nu = \frac{1}{4\pi} \int I_\nu(\Omega) d\Omega. \quad (42)$$

(In other words,  $J_\nu$  is just the angle-averaged version of  $I_\nu$ ).

- So far, we have assumed that only photons with energies that are **exactly**  $h\nu_{lu}$  can cause this transition to occur. In fact, this is an oversimplification, for reasons that we will come to shortly. We therefore write the absorption rate in the more general form

$$W_{lu} = \int_0^\infty B_{lu} J_\nu \phi_\nu d\nu, \quad (43)$$

where  $\phi_\nu$  is the line profile function, normalized such that  $\int_0^\infty \phi_\nu = 1$ .

- The rate at which energy is absorbed for photons in the frequency range  $\nu \rightarrow \nu + d\nu$  can therefore be written as:

$$P_{lu} = B_{lu} J_\nu h\nu \phi_\nu d\nu. \quad (44)$$

- The total amount of energy absorbed within our cylinder in a time element  $dt$  and within a solid angle element  $d\Omega$  is therefore

$$dE_\nu = \frac{1}{4\pi} n_l B_{lu} I_\nu(\Omega) h\nu \phi_\nu d\nu dV dt d\Omega, \quad (45)$$

where we have used the fact that

$$\frac{dJ_\nu}{d\Omega} = \frac{I_\nu(\Omega)}{4\pi}. \quad (46)$$

If we now compare equations 31 and 45, we see that

$$\alpha_\nu = \frac{h\nu}{4\pi} n_l B_{lu} \phi_\nu. \quad (47)$$

- Up to this point, we have only considered absorption. However, in real systems we also need to account for the effects of spontaneous and stimulated emission.
- Stimulated emission is typically accounted for by modifying our definition of the absorption coefficient, and treating stimulated emission as **negative absorption**. In place of Equation 47, we have:

$$\alpha_\nu = \frac{h\nu}{4\pi} (n_l B_{lu} - n_u B_{ul}) \phi_\nu. \quad (48)$$

The corresponding absorption cross-section is given by

$$\sigma_\nu = \frac{h\nu}{4\pi} (f_l B_{lu} - f_u B_{ul}) \phi_\nu, \quad (49)$$

where  $f_l \equiv n_l/n$  and  $f_u \equiv n_u/n$  are the fractional populations of the lower and upper levels, respectively.

- The stimulated emission term is unimportant when  $n_u \ll n_l$ , but becomes important once the upper and lower levels have comparable populations. If  $n_u > n_l$  – a situation known as a **population inversion** – then the absorption coefficient can become negative. In this case, Equation 37 shows us that we will get **exponential amplification** of our incident radiation. This is the principle behind the laser, and can also occur in astrophysical systems in the form of naturally occurring **masers**.
- To account for spontaneous emission, we define an emission coefficient  $j_\nu$ , such that the radiative energy emitted in a time  $dt$  in a direction  $\Omega$  with a frequency in the interval  $\nu \rightarrow \nu + d\nu$  from the particles within our cylindrical volume  $dV$  is given by

$$dE_{\nu,\text{emiss}}(\Omega) = j_\nu(\Omega) d\nu dV dt d\Omega. \quad (50)$$

With this definition, the change in the specific intensity over a distance  $ds$  due solely to spontaneous emission is given by  $dI_\nu = j_\nu ds$ . Therefore, the general transfer equation that accounts for both emission *and* absorption is written as

$$\frac{dI_\nu}{ds} = j_\nu - \alpha_\nu I_\nu. \quad (51)$$

- We can write the emission coefficient in terms of the Einstein coefficient  $A_{ul}$  as

$$j_\nu = n_u A_{ul} \frac{h\nu}{4\pi} \phi_\nu, \quad (52)$$

where  $\phi_\nu$  is the line profile function. Note that we have assumed that we can use the same line profile function for emission and absorption; this is a reasonable assumption at the level of detail that we'll be dealing with in these lectures.

- This transfer equation has the formal solution:

$$I_\nu(L) = I_{\nu,0}e^{-\tau_{\nu,L}} + \int_0^L j_\nu e^{-\tau_\nu(s)} ds, \quad (53)$$

where

$$\tau_{\nu,s} = \int_0^s \alpha_\nu(s') ds', \quad (54)$$

and where  $\tau_{\nu,L}$  is simply this quantity computed for  $s = L$ .

### 1.3.2 Kirchhoff's law

- Within a region that is in thermal equilibrium, the specific intensity of the radiation field is given everywhere by the Planck function

$$B_\nu(T) = \frac{2h\nu^3/c^2}{\exp\left(\frac{h\nu}{kT}\right) - 1}. \quad (55)$$

Because  $I_\nu$  does not vary spatially within this region,  $dI_\nu/ds = 0$  and hence Equation 51 becomes

$$j_\nu - \alpha_\nu B_\nu(T) = 0. \quad (56)$$

From this we see that in thermal equilibrium,

$$\frac{j_\nu}{\alpha_\nu} = B_\nu(T). \quad (57)$$

This relationship is known as **Kirchhoff's law**.

- We have derived Kirchhoff's law under the restrictive condition that the medium is in thermal equilibrium. However, the resulting law concerns a ratio of quantities that are determined solely by the level populations of our atoms and their Einstein coefficients, which do not depend (directly) on the radiation field. We therefore find that Kirchhoff's law holds whenever our atoms have the level populations that they would do if they were in full thermal equilibrium, a condition known as **local thermodynamic equilibrium** (LTE).
- If our medium is in LTE, we can therefore write the radiative transfer equation in the form

$$\frac{dI_\nu}{ds} = \alpha_\nu(s) [B_\nu(T(s)) - I_\nu(s)], \quad (58)$$

where the temperature  $T(s)$  can vary along the ray. We therefore see that the effect of absorption and emission is to drive  $I_\nu$  towards the local value of the Planck function.

### 1.3.3 The source function

- The result derived at the end of the last section suggests a useful formulation of the radiative transfer equation. We first define a quantity known as the **source function**:

$$S_\nu \equiv \frac{j_\nu}{\alpha_\nu}. \quad (59)$$

When our medium is in LTE,  $S_\nu = B_\nu(T)$ , but our definition above is valid even if we are not in LTE.

- We can use our previous results for  $\alpha_\nu$  and  $j_\nu$  to write  $S_\nu$  in terms of the Einstein coefficients and level populations:

$$S_\nu = \frac{n_u A_{ul}}{n_l B_{lu} - n_u B_{ul}}. \quad (60)$$

- If we rewrite the radiative transfer equation in terms of the source function, we find that

$$\frac{dI_\nu}{ds} = \alpha_\nu(s) [S_\nu(s) - I_\nu(s)]. \quad (61)$$

In other words, in the general case, the effect of absorptions and emissions within the gas is to drive  $I_\nu$  towards the source function.

- One important consequence of this is that if the medium is optically thick, with  $\tau_\nu \gg 1$ , and if the absorption and emission coefficients vary only slowly within the gas, then to a good approximation,  $I_\nu(s) \simeq S_\nu(s)$ .

### 1.3.4 Line profiles

- Spectral lines have a finite width for two important reasons. First, atomic energy levels are not infinitely sharp. Second, if we are dealing with absorption by a group of atoms, then their motions relative to each other mean that they will absorb at slightly different frequencies.
- During our discussion so far, we have behaved as though atomic energy levels are infinitely sharp, with every transition having a well-defined energy.
- In reality, this is an oversimplification. Any excited state has a finite lifetime, which we can write as  $\Delta t = A_{ul}^{-1}$  in the simple case of a two-level atom. This finite lifetime then implies that there is a limit to how accurately we can determine the level energy, set by the **Uncertainty principle**:

$$\Delta E \Delta t \geq \hbar. \quad (62)$$

In terms of the spontaneous transition rate, this gives us

$$\Delta E = A_{ul} \hbar, \quad (63)$$

which translates to an uncertainty in the frequency given by

$$\Delta\nu = \frac{A_{ul}}{2\pi}. \quad (64)$$

The broadening in the spectral line that results from this fundamental uncertainty is known as **natural broadening**.

- If there is more than one lower level that can be reached by a radiative transition from our upper level, then the parameter that determines the energy uncertainty associated with the upper level is the sum of the spontaneous radiative transition rates from the upper level to all accessible lower levels, i.e.

$$\Gamma = \sum_{l < u} A_{ul}. \quad (65)$$

- If we are interested in a transition between two excited states, then both will have an energy uncertainty associated with them. In this case, the appropriate value of  $\Gamma$  is the sum of the value for the two levels.
- Once we know  $\Gamma$ , we can write down the expression for the naturally broadened line profile as<sup>5</sup>

$$\phi_\nu = \frac{\Gamma / (4\pi^2)}{(\nu - \nu_0)^2 + [\Gamma / (4\pi)]^2}, \quad (66)$$

where  $\nu_0$  is the frequency at line-centre (i.e. the frequency that corresponds to the energy of the transition of interest).

- This profile is also known as a **Lorentzian** profile. It is sharply peaked around  $\nu = \nu_0$ , and falls off as  $\phi_\nu \propto \Delta\nu^{-2}$  for  $\Delta\nu \gg \Gamma/4\pi$ , where  $\Delta\nu \equiv |\nu - \nu_0|$ .
- Absorption and emission lines produced by a collection of atoms are also broadened because of the motion of the atoms with respect to each other, in a process known as **Doppler broadening**.
- Consider a small fluid element, consisting of enough atoms to allow us to treat emission and absorption as continuous rather than discrete processes, but small enough that we can ignore any velocity gradients across the fluid element.
- If we choose a frame of reference that is at rest with respect to the centre of mass of this fluid element, then we will see no bulk motion, but will still see the atoms moving with respect to us because of their thermal motions
- Suppose that in our rest frame, the frequency at the centre of a specified emission line is given by  $\nu_0$ . An atom moving relative to us with a line-of-sight velocity component  $v_r$  will emit instead at a frequency  $\nu_0(1 - v_r/c)$ , due to the Doppler effect.

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<sup>5</sup>Proof of this requires the tools of time-dependent perturbation theory, and is beyond the scope of this course.

- If we temporarily ignore the effects of natural broadening and treat the emission line produced by a single atom as a  $\delta$ -function, then we can write down the following expression for the line profile function produced by the motion of the atoms:

$$\phi_\nu = \int_{-\infty}^{\infty} f(v) \delta\left(\nu - \nu_0 \left\{1 - \frac{v}{c}\right\}\right) dv, \quad (67)$$

where  $f(v)$  is the velocity distribution function.

- If, as is usually the case, the atoms have a Maxwell-Boltzmann velocity distribution, then the line profile function becomes

$$\phi_\nu = \frac{1}{\sqrt{\pi} \Delta\nu_D} e^{-(\Delta\nu/\Delta\nu_D)^2}, \quad (68)$$

where  $\Delta\nu_D = b(\nu_0/c)$  is the **Doppler width** of the line and  $b^2 = 2kT/m$ , where  $m$  is the mass of the atom.

- This expression assumes that the motion of the atoms is purely thermal, but it can be extended to handle the case of “microturbulence” (i.e. turbulent motions with a coherence length much shorter than the size of our absorbing system) by adopting a value for  $b$  given by  $b^2 = 2kT/m + 2\sigma_{\text{mt}}^2$ , where  $\sigma_{\text{mt}}^2$  is the variance of the microturbulent velocity distribution.
- Equations 66 and 68 represent two limiting cases for the line profile function. If the natural line width,  $\Delta\nu_N \equiv \Gamma/4\pi$ , is much larger than the Doppler width, then the line will have a Lorentzian profile; on the other hand, if  $\Delta\nu_D \gg \Delta\nu_N$ , then the line will have a Doppler profile.
- Let’s quantify this for a strong emission line, e.g. Lyman- $\alpha$ . In this case, we have  $\Gamma = 6.3 \times 10^8 \text{ s}^{-1}$ , and hence  $\Delta\nu_N \simeq 5 \times 10^7 \text{ Hz}$ . If we have  $\Delta\nu_D = \Delta\nu_N$ , then this implies that  $b \simeq 5 \times 10^7 c/\nu_0$ , and since  $\nu_0 \simeq 2.47 \times 10^{15} \text{ Hz}$ , we find that  $b \simeq 2 \times 10^{-8} c \simeq 6 \times 10^{-3} \text{ km s}^{-1}$ . This is much smaller than the typical thermal velocity of a hydrogen atom in the ISM, and we therefore see that Doppler broadening generally dominates, at least for frequencies close to the line centre.
- Note, however, that the Doppler profile falls off exponentially with increasing  $\Delta\nu$ , unlike the Lorentzian profile, which falls off as  $\phi_\nu \propto \Delta\nu^{-2}$ . This means that at frequencies far from line center, the profile looks much more like a Lorentzian profile than like a Doppler profile. This gives rise to features known as the **damping wings** of the line.
- For reference, in the most general case in which both Doppler broadening and natural broadening are important, the line profile is given by the convolution of the two profiles, yielding a combined function known as the **Voigt profile**.

## 1.4 Collisional processes – a brief introduction

- So far, our atoms have interacted only with the radiation field, and not with each other. We now relax this assumption and consider the effect of collisions. These may be collisions between two atoms, or between an atom an elementary particle (proton, electron, etc.) or a molecule or ion.
- These collisions may change the quantum state of the atom; i.e. in the aftermath of the collision, it may be in a completely different quantum state compared to its state before the collision (provided that all applicable conservation laws are obeyed).
- We can describe collisions in a fully QM fashion using what is known as scattering theory. This is described in detail in references such as Atkins & Friedmann.
- However, this approach is highly mathematical and we don't really have time for it in this course. Instead, for the time being we will parameterize collision rates in terms of collisional cross-sections that we suppose we know how to calculate somehow. Later, we will see how we can use a quasi-classical approach to estimate these cross-sections.
- Let us start by considering the collisional rate coefficient  $q_{ij}^c$ . If the number density of atoms in some state  $|i\rangle$  is  $n_i$ , then the rate per unit volume with which these atoms undergo collisional transitions to some other state  $j$  owing to collisions with collisional partner  $c$  can be written as

$$C_{ij}^c n_i = q_{ij}^c n_c n_i. \quad (69)$$

- The total rate at which transitions between  $|i\rangle$  and  $|j\rangle$  occur due to collisions is then simply given by a sum over all possible collision partners:

$$C_{ij} = \sum_c C_{ij}^c. \quad (70)$$

Often, collisions with a single type of partner dominate this sum. For example, within the gas making up a giant molecular cloud, collisions with  $\text{H}_2$  typically dominate, and  $C_{ij} \simeq C_{ij}^{\text{H}_2}$ .

- The collisional rate coefficient  $q_{ij}^c$  can be written in terms of a collision cross-section  $\sigma_{ij}^c$ . As this cross-section is often a function of the velocity, it is necessary to integrate over the distribution of relative velocities,  $f(v)$ , and hence:

$$q_{ij}^c = \int_0^\infty \sigma_{ij}^c(v) v f(v) dv. \quad (71)$$

This velocity averaging is also sometimes indicated by the use of angular brackets, i.e. we write the integral above as  $\langle \sigma_{ij}^c v \rangle$ .

- Given a collisional rate coefficient  $q_{ij}^c$  for a transition from  $|i\rangle$  to  $|j\rangle$ , we can obtain the rate coefficient for the inverse process, a transition from  $|j\rangle$  to  $|i\rangle$ , by using the **principle of detailed balance**. This states that in thermal equilibrium, the rate at which collisions cause transitions from  $|i\rangle$  to  $|j\rangle$  must be the same as the rate

at which they cause transitions from  $|j\rangle$  to  $|i\rangle$ . It is a consequence of **microscopic reversibility**, i.e. the idea that the microscopic dynamics of particles and fields are time-reversible, because the governing equations are time-symmetric.

- In thermal equilibrium, we know that the ratio of atoms in state  $|j\rangle$  to those in state  $|i\rangle$  is simply given by the Boltzmann distribution:

$$\frac{n_j}{n_i} = \frac{g_j}{g_i} e^{-E_{ij}/kT}, \quad (72)$$

where  $g_i$  and  $g_j$  are the statistical weights of the two states and  $E_{ij}$  is the difference in their energies.

- The principle of detailed balance tells us that for any collisional transition, the following equation holds in thermal equilibrium:

$$q_{ij}^c n_i n_c = q_{ji}^c n_j n_c. \quad (73)$$

Rearranging this, we find that

$$q_{ij}^c = q_{ji}^c \frac{n_j}{n_i} \quad (74)$$

$$= q_{ji}^c \frac{g_j}{g_i} e^{-E_{ij}/kT}. \quad (75)$$

Alternatively, we can write  $q_{ji}^c$  in terms of  $q_{ij}^c$  as:

$$q_{ji}^c = \frac{g_i}{g_j} q_{ij}^c e^{+E_{ij}/kT}. \quad (76)$$

- Although we have assumed thermal equilibrium in order to derive this relationship, the final result is independent of the number of atoms in each state and hence holds even when the system is not in thermal equilibrium.

## 1.5 Radiative cooling – optically thin limit

- Consider a simple two-level atomic system, with lower level  $l$  and upper level  $u$ , separated by an energy  $E_{ul}$ . If the specific intensity of the radiation field at a frequency  $\nu_{ul} = E_{ul}/h$  is negligible, then we can write the rates of change of the level populations as:

$$\frac{dn_u}{dt} = C_{lu}n_l - C_{ul}n_u - A_{ul}n_u \quad (77)$$

$$\frac{dn_l}{dt} = -C_{lu}n_l + C_{ul}n_u + A_{ul}n_u \quad (78)$$

- Provided that the radiative and/or collisional transitions are rapid compared to any other timescales of interest in our problem, we can assume that the level populations will reach **statistical equilibrium**, in which case:

$$C_{lu}n_l = (C_{ul} + A_{ul})n_u. \quad (79)$$



- From our discussion of detailed balance above, we know that

$$\frac{C_{lu}}{C_{ul}} = \frac{g_u}{g_l} e^{-E_{ul}/kT}. \quad (80)$$

We can therefore write the ratio of the level populations as

$$\frac{n_u}{n_l} = \frac{(g_u/g_l)e^{-E_{ul}/kT}}{1 + A_{ul}/C_{ul}}. \quad (81)$$

In the limit that  $C_{ul} \gg A_{ul}$ , we see that we recover the Boltzmann distribution. On the other hand, in the limit that  $C_{ul} \ll A_{ul}$ , we find that

$$\frac{n_u}{n_l} \simeq \frac{C_{ul}}{A_{ul}} \frac{g_u}{g_l} e^{-E_{ul}/kT}, \quad (82)$$

or in other words that

$$\frac{n_u}{n_l} \simeq \frac{C_{ul}}{A_{ul}}. \quad (83)$$

- We see therefore that when collisions dominate over radiative decays, the level populations approach their LTE values, while in the other limit, collisional excitations are balanced by radiative de-excitations, and collisional de-excitations are unimportant.
- In the simple case in which collisions with a single species dominate  $C_{ul}$ , we can write the collisional de-excitation rate as  $C_{ul} = q_{ul}^c n^c$ , where  $n^c$  is the number density of the dominant collision partner. Since the key parameter that determines whether collisions or radiative decays dominate is the ratio  $A_{ul}/C_{ul}$ , we can define a **critical density** for our collision partner, such that this ratio is one:

$$n_{\text{cr}}^c \equiv \frac{A_{ul}}{q_{ul}^c}. \quad (84)$$

When  $n^c \gg n_{\text{cr}}^c$ , collisions dominate and the level populations tend to their LTE values. On the other hand, when  $n^c \ll n_{\text{cr}}^c$ , radiative decay dominates and most atoms are in their ground states.

- In the more general case in which collisions with several different species make comparably large contributions to  $C_{ul}$ , we can define the critical density in a more general fashion. If we take  $n$  to be some reference number density (e.g. the number density of H nuclei, which has the benefit that it is invariant to changes in the ratio of atomic to molecular hydrogen), then we can define a critical density with the following expression:

$$\frac{A_{ul}}{C_{ul}} \equiv \frac{n_{\text{cr}}}{n}. \quad (85)$$

Here,  $n_{\text{cr}}$  is the critical value of our reference density, rather than that of a specific collision partner. In terms of the individual fractional abundances and collisional de-excitation rates, we have:

$$n_{\text{cr}} = \frac{A_{ul}}{\sum_c q_{ul}^c x^c}, \quad (86)$$

where  $x^c \equiv n^c/n$ . Alternatively, if we divide through by  $A_{ul}$ , we can easily show that

$$n_{\text{cr}} = \left[ \sum_c \frac{x^c}{n_{\text{cr}}^c} \right]^{-1}, \quad (87)$$

where the critical densities for the individual colliders are given by Equation 84 above.

- Using our general definition of the critical density, we can write the ratio of the level populations of our two level atom as

$$\frac{n_u}{n_l} = \frac{(g_u/g_l)e^{-E_{ul}/kT}}{1 + n_{\text{cr}}/n}. \quad (88)$$

- We can now use the fact that for a two-level system,  $n_l + n_u = n_X$ , the total number density of our atoms, to rewrite this equation as

$$\frac{n_u}{n_X - n_u} = \frac{(g_u/g_l)e^{-E_{ul}/kT}}{1 + n_{\text{cr}}/n}. \quad (89)$$

- We can rearrange this to give

$$\frac{n_u}{n_X} = \frac{(g_u/g_l)e^{-E_{ul}/kT}}{1 + n_{\text{cr}}/n + (g_u/g_l)e^{-E_{ul}/kT}}. \quad (90)$$

In the limit that  $n \gg n_{\text{cr}}$ , we see immediately that we recover the Boltzmann distribution. On the other hand, in the limit that  $n \ll n_{\text{cr}}$ , the  $n_{\text{cr}}/n$  term in the denominator dominates and Equation 90 reduces to

$$\frac{n_u}{n_X} \simeq \frac{g_u}{g_l} e^{-E_{ul}/kT} \frac{n}{n_{\text{cr}}}. \quad (91)$$

We can then use the fact that for any collider,

$$\frac{q_{lu}}{q_{ul}} = \frac{g_u}{g_l} e^{-E_{ul}/kT}, \quad (92)$$

to write Equation 91 as

$$\frac{n_u}{n_X} \simeq \frac{\sum_c q_{lu}^c x^c}{A_{ul}} n. \quad (93)$$

- The power radiated by our collection of atoms can be written as

$$\Lambda_{ul} = A_{ul} E_{ul} n_u. \quad (94)$$

Physically, what is going on here is that we are converting energy from the thermal motion of the atoms and their collision partners into radiative energy (i.e. photons). The expression above therefore represents a **radiative cooling rate**.

- We can use Equation 90 to write this cooling rate in terms of our total number density of atoms,  $n_X$ . We find that

$$\Lambda_{ul} = A_{ul} E_{ul} n_X \frac{(g_u/g_l)e^{-E_{ul}/kT}}{1 + n_{cr}/n + (g_u/g_l)e^{-E_{ul}/kT}}. \quad (95)$$

- In the limit  $n \gg n_{cr}$ , which we will henceforth refer to as the LTE limit, we see that

$$\Lambda_{ul} = A_{ul} E_{ul} n_X \frac{(g_u/g_l)e^{-E_{ul}/kT}}{1 + (g_u/g_l)e^{-E_{ul}/kT}}, \quad (96)$$

$$= A_{ul} E_{ul} n_X \frac{(g_u/g_l)e^{-E_{ul}/kT}}{Z(T)}, \quad (97)$$

where in the second line, we have made use of the **partition function** for the atom. For a general multilevel system, this is defined as

$$Z(T) = \sum_i \frac{g_i}{g_0} e^{-E_{i0}/kT}, \quad (98)$$

where  $g_0$  is the statistical weight of the ground state,  $g_i$  is the statistical weight of excited state  $i$ ,  $E_{i0}$  is the energy difference between state  $i$  and the ground state, and where we sum over all possible excited states.

- Points to note about the LTE cooling rate:

- In the LTE limit, the cooling rate scales with the density as  $\Lambda_{ul} \propto n_X$ . In this limit, the *mean* cooling rate per atom depends solely on the temperature (and the properties of the atom itself), and the density dependence of the cooling rate therefore simply reflects how many atoms we have.
- The cooling rate is also directly dependent on the size of the Einstein coefficient,  $\Lambda_{ul} \propto A_{ul}$ . Therefore, stronger transitions provide much more cooling than weaker transitions.
- If we define a **cooling time**,

$$t_{cool} = \frac{\frac{1}{\gamma-1} n_{tot} kT}{\Lambda_{ul}}, \quad (99)$$

where  $\gamma$  is the adiabatic index of the gas, then in the LTE limit, this is *independent* of density.

- In the limit where  $n \ll n_{cr}$  (the low-density limit, or  $n \rightarrow 0$  limit), the cooling rate becomes

$$\Lambda_{ul} = E_{ul} \left( \sum_c q_{lu}^c n^c \right) n_X. \quad (100)$$

- Points to note about the low density cooling rate:

- At low densities, the cooling rate is *independent* of the value of  $A_{ul}$ : we get the same amount of cooling from either strong or weak transitions (provided that in both cases we have  $n \ll n_{\text{cr}}$ ).
- On the other hand, the cooling rate depends directly on the collisional excitation rate and hence on the number densities of the various colliders.
- This behaviour is easy to interpret: it occurs because in this limit, *every* collisional excitation is followed by radiative de-excitation and hence by the loss of a photon's worth of energy from the gas. Therefore, the factor controlling the cooling rate is how frequently the upper state can be excited, not how rapidly it decays once excited.
- This means that the overall density dependence of the low-density cooling rate is:  $\Lambda_{ul} \propto n^2$ . Consequently, the cooling time scales as  $t_{\text{cool}} \propto n^{-1}$ .

## 1.6 Radiative cooling – optical depth effects

- So far, we have completely ignored the effects of absorption. This is a reasonable assumption if any ambient radiation fields are very weak and the optical depth in the line is very small, but it does not hold in every case.
- Therefore, we now generalise our analysis to handle absorption and stimulated emission. Consider our two-level atom, with level populations that are in statistical equilibrium. In this case, we have:

$$(C_{lu} + B_{lu}J_{lu})n_l = (A_{ul} + B_{ul}J_{lu} + C_{ul})n_u, \quad (101)$$

where  $J_{lu}$  is the mean specific intensity.

- In general, to solve this equation throughout our medium, we need to know  $J_{lu}$  at every point, and since  $J_{lu}$  depends on the level populations, we end up with a nasty coupled problem that requires a numerical treatment to solve.
- However, there is an interesting limiting case which we can study here that is often applicable and that can help to illuminate the physics of the situation.
- We start by assuming that any incident radiation field is negligible, and hence that the only important contribution to  $J_{lu}$  comes from the emission of the atoms themselves. We also assume that photons that are emitted by the atoms either are absorbed locally (i.e. within a small volume around the emission site, within which we can assume that physical conditions such as density and temperature do not appreciably vary) or escape from the gas completely.
- The probability that a photon that is emitted at some point within a distribution of gas can escape from the gas without re-absorption – the so-called **escape probability**,  $\beta$  – will in general depend upon the mean optical depth at that point,  $\bar{\tau}_{ul}$ , i.e.  $\beta = \beta(\bar{\tau}_{ul})$ .

- We can use  $\beta$  to simplify Equation 101 by noting that the net number of absorptions (i.e. the number of photons absorbed minus the number produced by stimulated emission) must equal the number of photons emitted that do not escape from the gas, i.e.

$$(n_l B_{lu} - n_u B_{ul}) J_{lu} = n_u (1 - \beta(\bar{\tau}_{lu})) A_{ul}. \quad (102)$$

Using this, we can rewrite our equation for the statistical equilibrium level populations as

$$C_{lu} n_l = (C_{ul} + \beta(\bar{\tau}_{ul}) A_{ul}) n_u. \quad (103)$$

- In other words, in this approximation, the effect of the local absorptions is to reduce the effective size of the radiative de-excitation rate: we go from  $A_{ul}$  in the optically thin case to  $A'_{ul} = \beta(\bar{\tau}_{ul}) A_{ul}$  in the optically thick case. Therefore, all of our previously derived results still hold provided that we make the substitution  $A_{ul} \rightarrow A'_{ul}$ .
- One important consequence of this is that the critical density *decreases*: since  $n_{\text{cr}} \propto A_{ul}$ , we see that when the gas is optically thick,  $n_{\text{cr}} \propto \beta(\bar{\tau}_{ul})$ . This means that the effect of local absorption (also known as **photon trapping**) is to lower the density at which LTE is reached. The higher the optical depth, the more pronounced this effect becomes.
- The relationship between  $\beta$  and  $\tau_{ul}$  depends on the geometry and velocity structure of the gas, and for a completely general gas distribution may be difficult to determine. However, simple closed-form approximations exist for a number of simple geometries: for instance,

$$\beta(\tau_{ul}) \simeq \frac{1 - e^{-3\tau_{ul}}}{3\tau_{ul}} \quad (104)$$

for a plane-parallel slab.

## 1.7 Temperatures

- Before concluding our discussion of atomic level populations and related issues, it is useful to take a little time to discuss the different temperatures that one may encounter.
- The most familiar is the **kinetic temperature**, often referred to simply as ‘the temperature’, without a qualifier. For a gas with a thermal (i.e. Maxwell-Boltzmann) small-scale velocity distribution, the kinetic temperature is the parameter  $T$  appearing in the Maxwell-Boltzmann distribution function, and is thus a measure of the thermal velocity of the atoms.
- However, there are two other important measures of temperature that are commonly encountered in the study of the ISM: the **excitation temperature** and the **brightness temperature**.

### 1.7.1 Excitation temperature

- When our atoms are not in LTE, a convenient way to describe how close they are to LTE is the **excitation temperature**. For our model two-level atom, this is defined as:

$$T_{\text{ex}} \equiv \frac{E_{ul}}{k} \left[ \ln \left( \frac{n_l g_u}{n_u g_l} \right) \right]^{-1}. \quad (105)$$

- Re-arrangement of this expression yields

$$\frac{n_u}{n_l} = \frac{g_u}{g_l} e^{-E_{ul}/kT_{\text{ex}}}. \quad (106)$$

In other words, the excitation temperature is the temperature for which the Boltzmann distribution would yield the specified ratio of  $n_u$  to  $n_l$ .

- Clearly, when our atoms are in LTE,  $T_{\text{ex}} \equiv T$  – in other words, the excitation temperature is the same as the kinetic temperature of the gas.
- More generally, we can show that

$$\frac{T}{T_{\text{ex}}} - 1 = \frac{kT}{E_{ul}} \ln \left( 1 + \frac{n_{\text{cr}}}{n} \right), \quad (107)$$

demonstrating that when  $n \ll n_{\text{cr}}$ , we have  $T_{\text{ex}} \ll T$ .

- We can also write the cooling function for our two-level atom in terms of the excitation temperature. We know that

$$\Lambda_{ul} = A_{ul} E_{ul} n_u, \quad (108)$$

and from the definition of the excitation temperature, we also know that

$$n_u = \frac{g_u}{g_l} e^{-E_{ul}/kT_{\text{ex}}} n_l. \quad (109)$$

Also, we know that  $n_l = n_X - n_u$ , where  $n_X$  is our atomic number density. Therefore,

$$n_u = \frac{(g_u/g_l) e^{-E_{ul}/kT_{\text{ex}}}}{Z(T_{\text{ex}})} n_X, \quad (110)$$

where  $Z(T)$  is the partition function. The cooling rate is therefore simply given by

$$\Lambda_{ul} = A_{ul} E_{ul} n_X \frac{(g_u/g_l) e^{-E_{ul}/kT_{\text{ex}}}}{Z(T_{\text{ex}})}. \quad (111)$$

Comparing this with Equation 97, we see that  $\Lambda_{ul}(n, T) \equiv \Lambda_{ul, \text{LTE}}(n, T_{\text{ex}})$ ; in other words, the cooling rate is the same as it would be if the gas were in LTE with a temperature  $T_{\text{ex}}$ .

### 1.7.2 Brightness temperature

- The **brightness temperature** of a radiation field is the temperature that a Planck function would have to have in order to produce the same specific intensity  $I_\nu$  as the radiation field in question. In other words,

$$I_\nu = \frac{2h\nu^3/c^2}{e^{h\nu/kT_b} - 1}, \quad (112)$$

where  $T_b$  is the brightness temperature.

- Rearrangement of this equation yields the following expression for  $T_b$ :

$$T_b^{-1} = \frac{k}{h\nu} \ln \left[ 1 + \frac{2h\nu^3}{c^2 I_\nu} \right]. \quad (113)$$

- At long wavelengths, in the regime where  $h\nu \ll kT$ , we can approximate the Planck function as

$$B_\nu(T) \simeq \frac{2kT\nu^2}{c^2}. \quad (114)$$

(This is known as the **Rayleigh-Jeans limit**). In this regime, the brightness temperature is directly proportional to  $I_\nu$ :

$$T_b = \frac{c^2}{2k\nu^2} I_\nu. \quad (115)$$

- This formulation of  $T_b$  is a very good approximation for something like 21 cm radiation, where the energy of the transition corresponds to a temperature of order 0.07 K. It's use becomes increasingly questionable as we move to shorter wavelengths, however, and so we typically use this approximation only for lines with wavelengths  $\lambda > 1$  mm.
- Suppose we're interested in the brightness temperature of some particular emission line produced by a cloud of gas. If the optical depth of the line is large, then we know that the emergent specific intensity is simply given by the source function,  $S_\nu$ . If the excitation temperature of the cloud is reasonably uniform, then  $S_\nu \simeq B_\nu(T_{\text{ex}})$ , and hence  $T_b \simeq T_{\text{ex}}$ .
- If we add the additional assumption that the atoms are in LTE, then we know that  $T_{\text{ex}} \simeq T$ , and hence  $T_b \simeq T$ . In other words, the brightness temperature of a line is a good diagnostic of the gas temperature provided that
  - The line is optically thick,  $\tau \gg 1$ .
  - The level populations producing this particular line are in LTE (i.e.  $n \gg \beta(\tau)n_{\text{crit}}$ ).