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ASTRONOMY AND ASTROPHYSICS

Acoustic Waves in the Solar Atmosphere

VI. Feautrier Type Radiation Treatment

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Summary. A differential equation method to solve the radiative transfer equation in the presence of shocks is developed and tested against exact solutions. The method appears sufficiently accurate for cases where the shock is at relatively small optical depth. A comparison with an integral method shows that the present method is 5 times faster.

Key words: radiative transfer – acoustic waves – stellar chromospheres

1. Introduction

In Paper II (Kalkofen and Ulmschneider, 1977) as well as Paper V (Ulmschneider et al., 1978, Appendix A) of this series we have described a method of solution of the radiative transfer equation in the presence of shocks which was based on an integral representation. Although that method being exact for parabolic source functions is very accurate and well behaved it is rather slow when used on the computer due to many exponential functions and extensive expansions. This is not a great disadvantage if only one frequency point (grey atmosphere) and one angle point is taken as has been the case in our previous work. However, in future work usage of several frequency and angle points will be unavoidable. It consequently is of great importance to look for a faster method that prevents that the computer spends most of its time solving the transfer equation.

In the present work similar to Tscharnuter (1977) and Mihalas (1980) we describe a method of solution of the transfer equation in the presence of shocks that is a modification of the Feautrier method (1964, 1965) and is based on differential equations. Section 2 describes the method while Sect. 3 gives tests and a discussion.

2. Method

Let I^+ and I^- respectively be the outgoing and ingoing specific intensity along a ray inclined by an angle ϑ $(0 \leqq \vartheta \leqq \frac{\pi}{2})$ against the outward vertical in a stellar atmosphere. $\mu = \cos\vartheta \geqq 0$ is the angle cosine. τ^* and τ respectively represent the optical depth in the atmosphere and along the inclined ray. If we assume a symmetric source function $S(\mu) = S(-\mu)$ then the transfer equations may be written with

$$\tau = \frac{\tau^*}{\mu}, \tag{1}$$

$$\frac{dI^+}{d\tau} = I^+ - S, \tag{2}$$

$$-\frac{dI^-}{d\tau} = I^- - S. \tag{3}$$

Defining a mean intensity

$$J \equiv \frac{I^+ + I^-}{2} \tag{4}$$

and a flux

$$F \equiv 2\pi\mu(I^+ - I^-), \tag{5}$$

we find by subtracting and adding Eqs. (2) and (3)

$$\frac{dJ}{d\tau} = \frac{F}{4\pi\mu} = I^+ - J = J - I^- \tag{6}$$

and

$$\frac{dF}{d\tau} = 4\pi\mu(J - S). \tag{7}$$

Differentiating Eq. (6) we have Feautrier's differential equation

$$\frac{d^2J}{d\tau^2} = J - S. \tag{8}$$

Assuming a τ grid (τ_i, $i = 1, \ldots, N$) we now discretize Eqs. (6) and (8). Following Paper II in order to improve accuracy for large depths where J approaches S we write the difference equations in terms of $J - S$. From Eq. (8) we find with $\delta_i^j \equiv \tau_j - \tau_i$

$$\frac{(J-S)_{i-1}}{\delta_{i-1}^i \delta_{i-1}^{i+1}} - (J-S)_i\left(\frac{1}{2} + \frac{1}{\delta_i^{i+1}\delta_{i-1}^{i+1}} + \frac{1}{\delta_{i-1}^i \delta_{i-1}^{i+1}}\right) + \frac{(J-S)_{i+1}}{\delta_i^{i+1}\delta_{i-1}^{i+1}}$$
$$= \frac{(S_i - S_{i+1})}{\delta_i^{i+1}\delta_{i-1}^{i+1}} + \frac{(S_i - S_{i-1})}{\delta_{i-1}^i \delta_{i-1}^{i+1}} \tag{9}$$

for every interior point $i \neq 1, \neq N$. For the boundary points $i = 1, N$ we follow Auer (1967) and obtain

$$J_2 = J_1 + (J_1 - I_1^-)\delta_1^2 + \tfrac{1}{2}(J-S)_1(\delta_1^2)^2, \tag{10}$$

from which we have

$$(J-S)_2 = (J-S)_1(1 + \delta_1^2 + \tfrac{1}{2}(\delta_1^2)^2) - (I_1^- - S_1)\delta_1^2 + S_1 - S_2 \tag{11}$$

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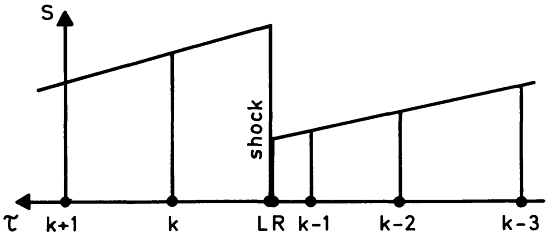

Fig. 1. Source function near a shock

and

$$J_{N-1}=J_N-(I_N^+-J_N)\delta_{N-1}^N+\tfrac{1}{2}(J-S)_N(\delta_{N-1}^N)^2\,, \tag{12}$$

from which we have

$$(J-S)_{N-1}=(J-S)_N(1+\delta_{N-1}^N+\tfrac{1}{2}(\delta_{N-1}^N)^2)$$
$$-(I_N^+-S_N)\delta_{N-1}^N+S_N-S_{N-1}\,. \tag{13}$$

For cases where no shock discontinuities occur Eqs. (9), (11), and (13) can be solved with recursion relations similar to the standard Feautrier fashion provided the boundary values $I_1^- - S_1$ and $I_N^+ - S_N$ are specified.

In the presence of shocks we have a discontinuous source function S but a continuous mean intensity J when going across the shock front.

Consider in Fig. 1 the source function near a shock. The shock discontinuity is between the infinitesimally close points L and R. These points e.g. lie between grid points k and $k-1$ of a given τ grid. In order to avoid complications when shock points lie close to grid points we follow the method described in Paper V, omit the grid point closest to the shock and replace it by the shock point. After completion of the solution in the thus modified grid we subsequently evaluate the solution at the disregarded grid point.

Let us consider the case where grid point $k-1$ is omitted. The case where point k is disregarded is treated analogously. Originally we followed Auer (1967) using Taylor expansions to second order similar to Eqs. (10) and (12) around the shock points L, R. The resulting solutions however were rather inaccurate as for intermediate optical depth intervals both $(\delta_i^{i+1})^2$ and $(J-S)$ were large such that the third term in the Taylor expansion was not small against the second. Note that this does not happen in Auer's case as at the boundary points either $(\delta_i^{i+1})^2$ or $(J-S)$ is very small. We thus discretize the first order Eqs. (6) at the shock points L, R:

$$\frac{J_k-J_L}{\tau_k-\tau_L}-J_L=I_L^-\,, \tag{14}$$

$$\frac{J_{k-2}-J_R}{\tau_{k-2}-\tau_R}-J_R=I_R^-\,. \tag{15}$$

As at the shock

$$I_L^-=I_R^-\,,\quad I_L^+=I_R^+\,,\quad J_L=J_R\,,\quad \tau_L=\tau_R\,, \tag{16}$$

we have

$$\frac{J_k-J_R}{\tau_k-\tau_R}=\frac{J_{k-2}-J_R}{\tau_{k-2}-\tau_R}\,, \tag{17}$$

from which we find

$$(J-S)_{k-2}\frac{1}{\tau_R-\tau_{k-2}}-(J-S)_R\left(\frac{1}{\tau_R-\tau_{k-2}}+\frac{1}{\tau_k-\tau_R}\right)+(J-S)_k\frac{1}{\tau_k-\tau_R}$$
$$=\frac{S_R-S_{k-2}}{\tau_R-\tau_{k-2}}+\frac{S_R-S_k}{\tau_k-\tau_R}\,. \tag{18}$$

Here the point R at the shock front takes the place of grid point $k-1$ and the solution is carried out in the usual Feautrier manner except that for the point R now Eq. (18) is used instead of Eq. (9) and appropriate differences δ are taken. After completion of the solution we find for the back of the shock

$$(J-S)_L=(J-S)_R+S_R-S_L\,. \tag{19}$$

The solution at the omitted grid point $k-1$ is obtained by using Eq. (6):

$$J_{k-1}=J_R+(J_R-I_R^-)(\tau_{k-1}-\tau_R)\,. \tag{20}$$

With Eqs. (15) and (16) we find

$$J_{k-1}=J_R+(J_k-J_R)\frac{\tau_{k-1}-\tau_R}{\tau_k-\tau_R}\,, \tag{21}$$

from which we have

$$(J-S)_{k-1}=(J-S)_R\left(1+\frac{\tau_R-\tau_{k-1}}{\tau_k-\tau_R}\right)-S_k+S_R$$
$$+\frac{\tau_R-\tau_{k-1}}{\tau_k-\tau_R}(S_R-S_k-(J-S)_k)\,. \tag{22}$$

This procedure is applicable for all situations except cases where shocks between grid points 1 and 2 or $N-1$ and N are very close to the boundaries. In these cases we omit the boundary points in favour of the shock point R. Using Eqs. (33)–(36) of Paper V we evaluate e.g. $(I^- - S)_R$ from the boundary values $(I^- - S)_1$. After completion of the solution we use those equations again to evaluate the solution $(J-S)_1$ at the disregarded boundary point.

3. Results and Discussion

After this work was completed the recent paper of Mihalas (1980) came to our attention. To effect a useful comparison between his and our method we have decided to adopt similar test situations in which we check our results against exact solutions. To avoid adding numerical errors arising from a double-Gauss quadrature scheme which is unrelated to the solution method of the transfer equation we however did not carry out angle integrations.

Following Mihalas we tested our method using linear source functions.

$$S(\tau)=\begin{cases}\alpha_1+\beta_1\tau & \text{for} \quad \tau\leqq\tau_S\\ \alpha_2+\beta_2\tau & \text{for} \quad \tau\geqq\tau_S\end{cases}, \tag{23}$$

where τ_S is the optical depth of the shock along the ray and where $\alpha_1=\beta_1=1$, $\alpha_2=10$, $\beta_2=\beta_1$. We use a τ^* grid extending from 10^{-4} to 10^2 with 8 or 15 intervals per decade.

The exact solution for the mean intensity is then

$$J_{\text{Exact}}(\tau)=\begin{cases}\alpha_1+\beta_1\tau+\dfrac{\alpha_2-\alpha_1}{2}e^{-(\tau_S-\tau)} & \text{for} \quad \tau\leqq\tau_S\\ \alpha_2+\beta_2\tau-\dfrac{\alpha_2-\alpha_1}{2}e^{-(\tau-\tau_S)} & \text{for} \quad \tau\geqq\tau_S\,.\end{cases} \tag{24}$$

Table 1. Maximum relative errors E for 8 and 15 intervals per decade as function of different shock position τ_S^*. Values in brackets are for cases where instead of Eq. (18) an Auer expansion was made. S marks cases where the maximum error occurs at the shock

τ_S^*	8 per decade		15 per decade	
10^{-2}	$2.2\ 10^{-3}$	$(3.9\ 10^{-3})$	$6.1\ 10^{-4}$	$(1.6\ 10^{-3})$
10^{-1}	$1.8\ 10^{-3}$	$(2.0\ 10^{-2})$	$5.1\ 10^{-4}$	$(1.1\ 10^{-2})$
1	$S\ 1.4\ 10^{-2}$	$S\ (1.4\ 10^{-1})$	$S\ 3.6\ 10^{-3}$	$S\ (7.9\ 10^{-2})$
10	$S\ 2.5\ 10^{-2}$	$S\ (1.8\ 10^{-1})$	$S\ 1.1\ 10^{-2}$	$S\ (1.6\ 10^{-1})$

We measure the accuracy of our method by computing the relative error $E=|J-J_{\mathrm{Exact}}|/J_{\mathrm{Exact}}$. Table 1 shows the maximum relative errors E for various cases τ_S^*.

The errors generally are in the same order of magnitude as those reported by Mihalas (1980). An exact comparison however has not been performed as Mihalas did not treat the one ray case. Nevertheless it appears that our method is similar if not superior to the cases using Hermitian equations. For cases where the shock is at low optical depth $\tau_S^* < 0.1$ which are the most interesting situations for stellar chromospheres the present method seems sufficiently accurate.

Table 1 clearly shows that usage of an Auer (1967) expansion to second order instead of Eq. (18) is considerably worse. This is due to the fact that the second order term in the Auer expansion is not small any more at the shock. The errors of Table 1 do not depend on the word length of the computer as tests with double precision did not improve the accuracy.

Calculations of the same cases as shown in Table 1 have been performed with the integral method described in Papers II and V. Here machine accuracy ($E \simeq 10^{-10}$) has been found in every case and at every depth. This is not surprising as that method was designed to give machine accuracy for parabolic source functions. It was found however that the integral method is about a factor of 5 slower in computation time than the present method. The overall increase of speed of the solution of the hydrodynamic code with the two stream approximation was a factor of about 1.5.

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