Propagation of nonlinear, radiatively damped longitudinal waves along magnetic flux tubes in the solar atmosphere

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Summary. For solar magnetic flux tubes we compare three types of waves: Longitudinal MHD tube waves, acoustic tube waves propagating in the same tube geometry but with rigid walls and ordinary acoustic waves in plane geometry. We find that the effect of the distensibility of the tube is small and that longitudinal waves are essentially acoustic tube waves. Due to the tube geometry there is considerable difference between longitudinal waves or acoustic tube waves and ordinary acoustic waves. Longitudinal waves as well as acoustic tube waves show a smaller amplitude growth, larger shock formation heights, smaller mean chromospheric temperature but a steeper dependence of the temperature gradient on wave period.

Key words: magnetic flux tubes – waves – chromosphere

1. Introduction

In this paper we consider the nonlinear, radiatively damped propagation of magnetoacoustic waves in a thin, intense magnetic flux tube embedded in the otherwise field-free solar atmosphere. Magnetoacoustic wave propagation in magnetic flux tubes in the last years has been extensively studied (Defouw, 1976; Roberts and Webb, 1978, 1979; Wilson, 1979, 1980, 1981; Parker, 1979; Wentzel, 1979; Spruit, 1981a, b, 1982; Roberts, 1983; Edwin and Roberts, 1983) particularly in the case of negligible gravity. These treatments usually start from the linearized MHD equations valid for tube geometries. In the case of zero gravity, the wave equation is solved in terms of a wave moving along the vertically-directed tube axis, where the horizontal variation is described in terms of Bessel functions both inside and outside the tube. Suitable matching at the tube boundary allows the construction of the complete wave. There are several wave types possible (see Spruit, 1982; Edwin and Roberts, 1983): a torsional mode, the nonlinear propagation of which has been studied by Hollweg et al. (1982); two transverse modes (also called taut wire or kink mode) and a longitudinal mode (also called axisymmetric or sausage mode), which is the subject of our present study.

Because the longitudinal mode is compressive, it closely resembles acoustic tube waves travelling in rigid funnels and ordinary acoustic waves in its propagation characteristics. For situations of non-negligible gravity, these waves have first been studied by Defouw (1976), who assumed an isothermal atmosphere. This study was subsequently generalized by Roberts and Webb (1978) to non-isothermal atmospheres. Because the pressure perturbation, and thus the wave in the medium outside the flux tube, was neglected to make the problem tractable, both studies are not as general as the previous work on cases of negligible gravity. This approximation was justified by the fact that for cases of negligible gravity and fully built-up external wave fields no mechanical energy is radiated away from the flux tube in horizontal direction if the external sound speed is larger than the tube speed (Spruit, 1982; see also Spruit and Zweibel, 1979). Thus the assumption of a constant pressure boundary which prevents transmission of acoustic energy seems a good approximation for this case. However, in highly nonlinear cases, where the wave amplitudes are not small, and particularly in situations where we have switch-on effects, the interfering wave field outside must be built up by transfer of wave energy to the outside medium; therefore some acoustic damping must occur. This question will be addressed in a separate paper.

Another effect which cannot be described by a linear treatment is shock formation. This is especially interesting for the problem of chromospheric and coronal heating. The heating mechanism for the outer atmosphere of stars is still unknown, but observations of solar and stellar UV and X-radiation fluxes (e.g. Linsky, 1980; Vaiana et al., 1981) have shown that a purely acoustic wave heating theory cannot be correct, and that there is good correlation between heating and the presence of magnetic fields. Because much of the observed UV and X-ray variation from star to star can be explained by different magnetic field coverage and because hydrodynamic shock formation nicely explains the sudden onset of chromospheric radiation loss (Ulmschneider and Stein, 1982) as derived from empirical solar models, it seems appropriate to explore the propagation of compressive modes in the presence of structured magnetic fields.

A systematic comparison of longitudinal tube waves, acoustic tube waves and ordinary acoustic waves in plane geometry has been made by Rae and Roberts (1982) and by Roberts (1981, 1983) in the linear regime. In the present work we compare these three wave types in the nonlinear regime. In particular we are interested in the shock formation properties to explore whether longitudinal tube waves could be a chromospheric heating mechanism.

From purely acoustic work (Ulmschneider et al., 1978; henceforth USKB78) it is well-known that radiation damping is important in realistic situations. Webb and Roberts (1980) and Roberts (1983) have in fact computed linear tube waves with constant radiative relaxation time $\tau_R$. Although this gives a basic feeling for the properties of radiatively damped longitudinal waves, it is also known that $\tau_R$ is a rapidly increasing function of height (Ulmschneider, 1971). In a time-dependent nonlinear
Equations (4) is the energy equation written in the form of an entropy conservation law. Here $S$ is the entropy per gram. For the radiation damping function $\frac{dS}{dt}_{\text{Red}}$ we use a grey LTE approximation (Ulmschneider et al., 1978; henceforth USK 78):

$$\frac{dS}{dt}_{\text{Red}} = \frac{4\pi \kappa}{T} (J - \bar{B}),$$

(5)

where

$$\kappa = 1.38 \times 10^{-23} \rho^{0.74} T^4$$

(6)

is the grey Rosseland opacity per gram fitted to calculations of Kurucz (1979). $T$ is the temperature within the tube which is related to $p$ and $\rho$ over the ideal gas equation

$$\rho = \frac{8RT}{\mu}$$

(7)

where $8R = 8.3144 \times 10^7$ erg/Kmol is the gas constant and $\mu = 1.3$ g/mol is the mean molecular weight for a non-ionized gas.

$$J = \frac{\sigma}{\pi} T_\epsilon (z)^4$$

(8)

is the adopted mean intensity, $T_\epsilon$ is the temperature outside the tube, $\sigma$ the Stefan-Boltzmann constant and

$$\bar{B} = \frac{\sigma}{\pi} T^4$$

(9)

is the Planck function. In this treatment of the radiative transfer we assume that the flux tube is optically thin and that the tube is embedded in a radiation field dominated by the external medium. A comparison with the radiation treatments of Webb and Roberts (1980) and of Roberts (1983) who use Newtons law of cooling and a constant radiative relaxation time is given in Appendix B.

2.2. Acoustic tube wave equations

It is instructive to compare the longitudinal tube wave Eqs. (1) to (9) with the similar equations of acoustic waves in a tube with rigid walls of the same geometry. Let $A$ be the cross section of the tube and $a$, the geometrical distance along the tube axis at some instant $t_0$. This distance $a$ is called Lagrange coordinate. If the Eulerian coordinate $z(a,t)$ along the tube is assumed to be a function of the Lagrangian coordinate $a$ and time $t$, the conservation of mass in the tube can be written in the form (index 0 labels quantities at time $t_0$)

$$\rho(a,t) A(a,t) \frac{dz}{dt} = \rho_0 A_0 \frac{dz}{dt}$$

(10)

or

$$\frac{\partial z}{\partial a} \rho A_0 = \frac{\partial z}{\partial a} \rho A.$$

(11)

Differentiating Eq. (11) with respect to $t$ and using

$$u = \frac{\partial z}{\partial t}$$

(12)

we have the continuity equation in Lagrangian form

$$\left( \frac{\partial \rho A}{\partial t} \right)_a + \frac{\partial^2 A^2}{\partial a^2} \left( \frac{\partial u}{\partial a} \right)_a = 0.$$

(13)
With the usual relations

\[
\left( \frac{\partial f}{\partial t} \right)_a + \left( \frac{\partial f}{\partial z} \right)_a = 0,
\]

(14)

\[
\left( \frac{\partial f}{\partial t} \right)_{t} + \left( \frac{\partial f}{\partial z} \right)_{t} = 0,
\]

(15)

used when going over to the Eulerian frame we find from Eq. (13) the Eulerian continuity equation for acoustic tube waves

\[
\left( \frac{\partial A}{\partial t} \right)_s + \left( \frac{\partial qA}{\partial z} \right)_t = 0.
\]

(16)

This equation is well-known in the literature of engineering aerodynamics and duct flow (e.g. Rudinger, 1969, p. 12; Lighthill, 1980, p. 92). In plane parallel geometry, \( A \) is constant and Eq. (16) reduces to the familiar continuity equation (Landau and Lifshitz, 1959, p. 2). The momentum and energy equations for tube waves in Lagrangian form can be written as

\[
\left( \frac{\partial A}{\partial t} \right)_a + A \left( \frac{\partial p}{\partial A} \right)_a + qA_A \theta = 0,
\]

(17)

and

\[
\left( \frac{\partial S}{\partial t} \right)_A = \frac{dS}{dt} \bigg| \text{Rad}.
\]

(18)

Here \( A_A \) is the mass element (g/cm); the pressure force acts on the actual cross section \( A \). With Eqs. (14) and (15) it is easily seen that the Eulerian forms of (17) and (18) are identical to Eqs. (2) and (4).

2.3 Differences between longitudinal and acoustic tube waves

As the magnetic flux in a flux tube is concerned we have

\[
BA = \Phi = \text{const}.
\]

(19)

With this relation it is seen that the continuity equations (1) and (16) are identical; and similarly for the momentum and energy equations. However, because \( B \) can be eliminated from Eq. (3) in favour of \( A \), it is clear that longitudinal MHD tube waves are acoustic tube waves with a specific "breathing law", which specifies the variation of the tube cross section with gas pressure. The distensibility of the tube adds an additional restoring force to the volume compressibility and thus alters the propagation speed of the wave (Lighthill, 1980, p. 93). Note that this distensibility is not a wall effect like in human blood vessels but is a volume effect. For the propagation speed in the longitudinal tube wave one has from Eqs. (3), (19)

\[
c^2_A = \frac{1}{\rho(\kappa + 2\theta) + \frac{1}{\rho} \left( \frac{1}{V} \frac{\partial V}{\partial p} + \frac{1}{A} \frac{\partial A}{\partial p} \right)} = \frac{c^2_A c^4_A}{c^2_A + c^4_A}.
\]

(20)

Here \( V = \rho^{-1} \) is the specific volume, \( \kappa \) the adiabatic compressibility, \( \theta \) the adiabatic distensibility, \( c_s \) the sound speed and \( c_A \) the Alfvén speed. It is seen that while in a rigid acoustic tube wave the propagation speed is the sound speed, the propagation speed \( c_A \) of the longitudinal tube wave is always less than both \( c_A \) and \( c_s \). In addition to the modification of the propagation speed, the ordinary acoustic wave, the acoustic tube wave and the longitudinal tube wave differ in the acoustic cut-off frequency \( \omega_c \). As this has already been found by Defovu (1976) and e.g. by Rae and Roberts (1982) we do not repeat a discussion here.

3. Shock waves

We now consider the formation of shocks as compressive tube waves propagate down a density gradient.

3.1. Hugoniot relations for longitudinal tube waves

Purely hydrodynamic shocks occur over distances of the order of the molecular mean free path

\[
I = \frac{1}{nq},
\]

(21)

which in the low chromosphere is about 10 cm. Here \( n \) is the number density of the gas and \( q \) is the hydrogen collision cross section, which in the low chromosphere can be written

\[
q = \pi(2a_0)^2,
\]

(22)

where \( a_0 \) is the Bohr radius. As the typical length scale for acoustic waves is of the order of the scale height, which is in the range of \( 10^7 \) cm, it is a good approximation to represent a hydrodynamic shock by a discontinuity. For longitudinal tube waves we now assume that the region where the strong bending of magnetic field lines near the hydrodynamic shock occurs is very small compared to the typical length scale of the acoustic waves, but is also considerably larger than \( I \). In addition, we disregard for the moment the occurrence of modecoupling in the near shock region; this will be discussed in a subsequent paper. The Hugoniot relations for longitudinal tube waves may then be derived as follows. Consider the shock discontinuity in a frame comoving with the shock (see Fig. 1). At the front the gas enters the shock with velocity \( v_1 \), density \( \rho_1 \) and pressure \( p_1 \). The tube cross section is \( A_1 \). At the back the gas leaves with velocity \( v_2 \), density \( \rho_2 \) and pressure \( p_2 \). The tube cross section is now \( A_2 \). Let us assume that the longitudinal tube wave equations are written in the frame comoving with the shock. We integrate the continuity Eq. (16) over the discontinuity from height \( z_1 \) to \( z_2 \). Take the time derivative of the first term out of the integral; because \( \rho A \) is bounded, we see that this integral vanishes for \( z_1 - z_2 \rightarrow 0 \). From the second term of Eq. (16) we get the first Hugoniot relation

\[
\rho_2 v_2 A_2 = \rho_1 v_1 A_1.
\]

(23)

Multiplying Eq. (16) with \( v \) and Eq. (2) with \( \rho A \) and adding, we have

\[
\frac{\partial}{\partial t} (\rho v A) + \frac{\partial}{\partial z} (\rho v^2 A) + A \frac{\partial p}{\partial z} + \rho A g = 0.
\]

(24)

From Eq. (3) and (19) we have

\[
A = \phi \left( 8 \pi (p_e - p) \right)^{-1/2},
\]

(25)

\[
\begin{array}{c}
A_1 \quad \text{shock} \\
| \quad | \quad | \quad | \\
T_1 & p_1 & g_1 & B_1 \\
| \quad | \quad | \\
T_2 & p_2 & g_2 & B_2 \\
\end{array}
\]

Fig. 1. Physical quantities in front of (labeled 1) and behind (labeled 2) the shock in a frame comoving with the shock.
where $U_{sh}$ is the shock velocity in the Eulerian frame. The shock velocity in the Lagrangian frame is, e.g.,

$$\left( \frac{\partial A}{\partial t} \right)_{sh} = \left( U_{sh} - u_1 \right) \frac{\partial A}{\partial z},$$

where 1 labels the region in front of the shock.

### 3.2. Hugoniot relations for acoustic tube waves

As the tube cross section does not change across the shock for acoustic tube waves in rigid tubes, the Hugoniot relations (23, 28, and 33) for these waves reduce to the ordinary Hugoniot relations (Landau and Lifshitz, 1959, p. 319). These relations can be obtained by going to the limit of very small distensibility, e.g., $\xi \ll 1$ where

$$\xi \equiv \frac{p}{p_e}.$$

From Eq. (25) we have to first order

$$A \approx \text{const} \, p_e^{1/2} (1 + \frac{1}{2} \xi),$$

which, when inserted in Eqs. (23) and (28), leads to the ordinary Hugoniot relations in the limit $\xi \to 0$.

### 3.3. Solution of the Hugoniot relations

In the time-dependent work described in the next section it is necessary to solve the Hugoniot relations a considerable number of times. It thus is important to develop an efficient method of solution. For this solution the physical variables $u_1, c_s, S_1, p_1, \xi_1, W_1, A_1, p_e$ in front of the shock are known in addition to the post shock velocity $u_2$. Together with the thermodynamic relations

$$W = \frac{c_s^2}{\gamma - 1},$$

$$\frac{c_s^3}{\gamma - 1} = \frac{p}{\rho},$$

and

$$\frac{\theta_2}{\theta_1} = \left( \frac{c_s}{c_s} \right)^{\frac{2}{\gamma - 1}} e^{-\frac{\mu \delta}{\theta}},$$

valid for a non-ionizing ideal gas, Eqs. (23), (25), (28), and (33)–(35) are sufficient to determine the 9 unknowns $c_s, S_2, p_2, \xi_2, W_2, A_2, v_1, v_2, U_{sh}$. Eliminating all unknowns except $A_2$, one is able to derive a third order equation for $A_2$,

$$E_0 + E_1 A_2 + E_2 A_2^2 + E_3 A_2^3 = 0,$$

where, by defining $\mu u \equiv u_2 - u_1$, one has

$$E_0 = \frac{\gamma - 2}{\gamma - 1} \Delta u^2 c_s^2 A_1^4,$$

$$E_1 = \Delta u^2 c_s^2 A_1^3 \left[ \frac{4 - 3\gamma}{\gamma - 1} \frac{4\pi}{\phi^2} \Delta u^2 c_s^2 \xi_1 \frac{A_1^2}{\gamma - 1} - \frac{8\pi}{\phi^2} W_1 \xi_1 A_1^2 \right],$$

$$E_2 = 2 \Delta u^2 c_s^2 A_1^2 \left[ \frac{1 - 2\gamma}{\gamma - 1} \frac{4\pi}{\phi^2} \Delta u^2 c_s^2 \xi_1 A_1^2 + \frac{4\pi}{\phi^2} W_1 \xi_1 A_1^2 \right],$$

$$E_3 = \frac{\gamma}{\gamma - 1} \frac{8\pi}{\phi^2} \Delta u^2 p_e c_s^2 A_1^2 \left[ \frac{4\pi}{\phi^2} \Delta u^2 \xi_1 A_1^2 - 1 \right].$$
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Of the three solutions of Eqs. (42), the first is negative and the second is smaller than \( A_1 \). They are thus both unphysical, for the latter because of the pressure increase associated with the entropy jump. The correct solution is the largest root \( A_2 \) which is found by Newton's method. In cases where the shock is still weak, the two positive solutions are very close together but can be separated by determining the point of zero slope between the two solutions from Eq. (42). The remaining unknowns can be found by using

\[
\varrho_2 = \frac{\varrho_1 A_1}{A_2} \left[ \frac{A_1 - A_2}{4 \pi \frac{\partial \varrho}{\partial u} \varrho \varrho_1 A_1^2 A_2 + A_1 - A_2} \right],
\]

(47)

\[
v_1 = -\frac{\varrho_2 A_2 A u}{\varrho_2 A_2 - \varrho_1 A_1},
\]

(48)

\[
v_2 = -\frac{\varrho_1 A_1 A u}{\varrho_2 A_2 - \varrho_1 A_1},
\]

(49)

and Eqs. (25), (34), and (39)-(41).

4. Time-dependent non-linear calculations: method

The discussion of the solution of the linear longitudinal MHD tube wave equations has already been given by Defouw (1976), by Roberts and Webb (1978) and others. The purpose of our present work is to make a detailed comparison with acoustic tube waves as well as with ordinary acoustic waves both for linear and non-linear cases and to derive the phase relations. For this reason the linear solutions are discussed in the Appendix.

4.1. Initial atmosphere, flux tube model

Time-dependent calculations can start from any initial state consistent with the dynamical equations. As we want to study primarily the properties of longitudinal tube waves, and not the global behaviour of the solar atmosphere, it is therefore advantageous to start from an initial atmosphere which, in absence of waves, does not show time dependence. This is a radiative equilibrium atmosphere.

Following our previous work on acoustic waves (e.g., USKB 78) we adopt a grey radiative equilibrium atmosphere outside the flux tube. As the regions between the flux tubes may be quite cool (Ayres, 1981), such a model which neglects the chromospheric temperature rise seems quite appropriate for the external medium. With external pressure \( p_e \) and temperature \( T_e \) thus given as function of height, the time independent static versions of Eqs. (2), (3), (7), and (19) with \( T_0 = T_c \) can then be solved for \( B_0 \), \( p_0 \) and \( A_0 \) as functions of height if \( B_{00} \) and \( A_{00} \) are specified at the bottom of the flux tube. For the magnetic field strength \( B_0 \) and the tube cross section \( A_0 \) typical values are used which were determined empirically (Zwaan, 1978; Stenflo, 1978). As customary the zero height level \( z = 0 \) at the bottom of the tube is taken where the external atmosphere has optical depth unity at 500 nm. The non-uniform grid spacing \( \Delta z \) is chosen such that the wave is well-resolved, and such that there are enough height points per scale height (see Ulmschneider et al., 1977). The resulting initial flux tube model (dashed) is exhibited in Fig. 2, it is called model \( A \).

The crowding of the flux tubes at the (super-)granulation boundaries does not permit an exponential horizontal spreading of individual tubes to go on indefinitely. To take into account this effect we have constructed model \( B \) which above a height \( z_0 = 500 \) km spreads only linearly with height. As the grey radiative equilibrium atmosphere external to the flux tube has an isothermal boundary temperature with scale height \( H \) we assume using

\[
A_0 = A_{00} e^{-z/2H}
\]

(50)

for \( z > z_0 \) that the tube radius increases linearly as

\[
r = r_0 + \frac{r_0}{4H} (z - z_0)
\]

(51)

Fig. 2. Initial flux tube model \( A \) (dashed) and model \( B \) (solid). The magnetic field strength \( B_0 \) is given in (G), the gas pressure \( p \) in (dyn/cm²) and the density \( \varrho \) in (g/cm³). The other quantities are defined in the text. \( r \) shows the instantaneous tube radius of the wave of Fig. 4
from which the magnetic field strength $B_0$ can be calculated using Eq. (19). For $z > z_0$, the horizontal pressure balance Eq. (3) in the model construction is thus replaced by

$$p + \frac{\Phi^2}{8\pi^2 \left( \frac{r_0 + r_0}{4H} (z - z_0) \right)^4} = p_a(z)$$

(52)

from which a modified external pressure $p_a(z)$ is computed which supposedly takes into account external magnetic stresses. Model $B$ (drawn) is shown in Fig. 2. From this figure it is seen that the Alfvén speed $c_A$ is always a factor of between 2 and 3 greater than the sound velocity $c_s$; for this reason the tube speed $c_T$ closely follows $c_A$. The tube spreads from a radius of $r = 57$ km at height $z = 0$ to $r = 167$ km at $z = 500$ km, and $r = 849$ (663) km at $z = 900$ km; the value in brackets is for model $B$. In model $A$, the gas pressure $p_0$ inside the tube is reduced by a factor of 3.5 compared to $p_{0,0}$, while toward the top of model $B$ it decreases more rapidly.

4.2. Method of characteristics

In order to numerically solve the longitudinal tube wave equations we follow our work on acoustic waves (Ulmschneider et al., 1977), and select the method of characteristics. This method has been shown (Hammer and Ulmschneider, 1978) to be much more efficient than finite difference methods for problems of the sort we deal with here; in addition, it allows shock finding and a detailed treatment of shock. In this method, the system of three partial differential Eqs. (1), (2), (4) in Eulerian form, or (13), (17), (18) in Lagrangian form are written as a system of 6 ordinary differential equations, of which 3 equations describe the three characteristics ($C^+, C^-, C^0$). After some algebra similar to that for acoustic waves (Ulmschneider et al., 1977), we find for these ordinary differential equations in Lagrangian form

$$du = \frac{2}{\gamma - 1} c_S dS + \frac{\mu c^2}{\gamma \rho \rho'} ds = \left( \frac{\rho c_T}{\gamma \rho} (\gamma - 1) \right) \frac{ds}{dt}$$

$$+ \frac{\rho c_T}{\gamma \rho' \rho^2} \frac{dp_e}{dz} \left( \frac{\gamma}{\gamma - 1} \right) dt = 0$$

(53)

along the characteristics, which are given by

$$\frac{da}{dt} = \pm c_T \frac{Q A}{Q_0 A_0}$$

(54)

Here the upper sign is for the $C^+$, and the lower for the $C^-$ characteristic. The energy equation gives

$$dS = \frac{ds}{dt} \left|_{\text{rad}} \right. \int dt$$

(55)

along the $C^0$ characteristic, $a = \text{const.}$

For the case of acoustic tube waves with rigid walls we find instead

$$du = \frac{2}{\gamma - 1} c_S dS + \frac{\mu c_S}{\rho \rho'} ds = \left( \frac{\rho c_T}{\gamma \rho} (\gamma - 1) \right) \frac{ds}{dt} \left|_{\text{rad}} \right. \int dt$$

$$- \frac{\mu c_S}{\rho A_0} \frac{dA_0}{dz} \left( \frac{\gamma}{\gamma - 1} \right) \frac{ds}{dt} = 0$$

(56)

$$\frac{da}{dt} = \pm c_S \frac{Q A}{Q_0 A_0}$$

(57)

For constant $A$ these equations go over to the ordinary acoustic waves in gravitationally-stratified atmospheres. The details of the numerical solution procedure and boundary conditions are similar to the purely acoustic case, and are described by Ulmschneider et al. (1977).

4.3. Wave flux

As the energy flux in Eq. (32) is identical to that in ordinary acoustic waves, the wave flux can be similarly derived (Landau and Lifshitz, 1959, p. 251) as

$$F_M = \bar{p} \bar{u}$$

(58)

where the bar denotes a time mean and the prime the difference between the actual and the time averaged value. As for adiabatic waves, the total wave energy flux $A_0 F_M$ is conserved; we thus see from Eqs. (50), (A.2), and (A.4) that the exponential steepening behaviour of the solution is a consequence of energy conservation. For $\omega \gg \omega$, one finds from the phase relations (A.7), (A.8), and (A.10) that the wave amplitudes are connected by

$$\frac{p'}{\rho_0} = \frac{q}{\rho_0} \frac{u'}{\bar{u}} \approx \frac{q_{0,0} - \rho_0}{c_T^2} \frac{1}{c_0^2}$$

(59)

Thus in the limit $\omega \gg \omega$, the wave flux is given by

$$F_M = \rho_0 c_T u' \bar{u}$$

(60)

In the present work we used Eq. (60) and adopted a piston velocity

$$u = -u_0 \sin \left( \frac{2\pi}{P} i \right)$$

(61)

where $P$ is the wave period and the initial wave amplitude is

$$u_0 = \left( \frac{2 F_M}{\rho_0 c_T} \right)^{1/2}$$

(62)

Here $\rho_0$ and $c_T$ are values at the bottom of the flux tube.

5. Time-dependent non-linear calculations: results

In this section, we discuss the nature of the solutions obtained from our numerical simulations of compressible tube waves.

5.1. General properties

of radiatively damped longitudinal tube waves

Figures 3-6 show computed wave profiles obtained for cases of a fixed mechanical flux $F_M = 1.5 \times 10^8$ erg/cm$^2$/s and various wave periods $P = 20, 30, 45, 60$ s. For ready comparison these figures are plotted on the same scale. The flux tube is that of model $B$. A comparison with the purely acoustic wave computations of USKB 78 shows at first a strikingly similar overall behaviour, but there are interesting differences in the details. First, the velocity, $u$, pressure, $p$, and temperature, $T$, amplitudes increase much less steeply with height in the tube calculations, a result which is due to the radiation spreading [cf. Eqs. (A.2)-(A.4)]. Second, the amplitude of the radiation damping function $D = \frac{ds}{dt} \left|_{\text{rad}} \right.$ decreases much more rapidly with height in the tube calculations when compared to the results of USKB 78; this difference is due to the much lower gas pressure in the flux tube and to the fact that in the present results,
the weakly increasing temperature oscillations do not strongly oppose the rapid decay of the opacity with height. This effect is especially apparent at greater heights, where in spite of the $T^8$ temperature dependence [cf. Eqs. (6) and (9)] the radiative emission behind the shocks is greatly decreased compared to the USKB 78 results.

The phase shifts between the physical variables are similar to the pure acoustic case, and are dominated by the short period ($\omega \gg \omega_c$) behaviour. The oscillations of the pressure, $p$, velocity, $u$, and temperature, $T$, are in phase [cf. Eqs. (A.8)-(A.10)], while the damping function $D$ is $180^\circ$ out of phase with $T$, as is expected from Eqs. (5), (9) for an optically thin situation. The time-averaged
temperature, $T$, behaves quite similarly to the cases of USKB 78. At heights lower than the temperature minimum, the mean temperature is depressed below the initial radiative equilibrium temperature, $T_0$, due to the non-linearity of the Planck-function, as explained by USKB 78. Above the temperature minimum, shock dissipation raises the mean temperature significantly. However, due to the weakly increasing wave amplitude and consequently decreased dissipation rate, the chromospheric temperature rise is much smaller for longitudinal tube waves. This again is due to the spreading of the flux tube, which keeps the shock strength small.

The temperature minima in Figs. 3–6 occur at progressively greater heights as the wave period is increased. This is expected.
from the increase of the shock formation height, $z_{SB}$, with wave period (cf. Sect. 5.3, below). For short period waves the increase of $T$ due to shock dissipation occurs at low height where small temperature excesses generate significant radiation losses. Thus the 20 s wave spends most of its mechanical energy at low height. At greater height the tube spreading and the low mechanical flux lead to weak shocks and thus to only an insignificant chromospheric temperature rise. With larger wave period and therefore higher $z_{SB}$, the amount of mechanical energy available at greater heights is increased, leading to stronger shocks and consequently to steeper chromospheric temperature gradients (cf. Figs. 3–6).

5.2. Wave pressure, magnetic field perturbation and tube cross section

Another rather interesting quantity in the present calculation is the time-averaged pressure $\bar{p}$. Consider the case of hydrostatic equilibrium where the first term in Eq. (17) can be neglected and where after integrating over the appropriate atmospheric slab one has

$$\bar{p} = -\int_{\alpha}^{\infty} \frac{\theta_0 \Delta A}{A} d\alpha = -\int_{\alpha}^{\infty} \theta_0 \frac{d\alpha}{A} = p_0.$$  

(63)

This shows that in the case of hydrostatic equilibrium, $\bar{p}$ at a Lagrangian height always stays equal to the pressure of the initial atmosphere at that height. Here $\bar{p}$ is essentially gravity times the mass column density of the atmospheric slab above the considered height. In our present calculation, however, where hydrostatic equilibrium is replaced by the dynamical equilibrium, wave pressure augments gas pressure, and consequently $\bar{p} < p_0$ in the dynamically steady state. Moreover, the mass zones are displaced in the Eulerian frame as a consequence of heating and cooling processes. Thus, at low heights in the Eulerian frame the effects of the temperature depression and wave pressure combine to decrease $\bar{p}$ (cf. Figs. 3–6). At greater heights, shock heating expands the atmosphere and thus $\bar{p}$ becomes larger than the initial pressure at that height. This has also been found for purely acoustic waves (USKB 78).

The height dependence of $\bar{p}$ helps us to understand the behavior of the magnetic field oscillation $B'$. In Figs. 3–6 $B' = B(z, t) - B_0(a)$ is shown in the Eulerian frame; it is seen that $B'$ is nonsymmetric. This effect is explained by the behaviour of the time averaged pressure $\bar{p}$. At low heights where $\bar{p} < p_0$ the magnetic field after Eq. (3) oscillates around a value $\bar{B} = B_0$ while at greater heights where $\bar{p} > p_0$ one has $\bar{B} < B_0$. As $p$ and $u$ are correlated, the $B'$ oscillation is 180° out of phase with $u$.

Another interesting quantity is the tube cross section $A$. After Eq. (19) the oscillations in cross section, $A'$, and $B'$ are 180° out of phase with one another; but $A$ is in phase with $p$ (cf. Eq. (25)). The actual tube cross section as function of height for the same wave and instant of Fig. 4 is shown in Fig. 2. As expected it is seen that the tube cross section is always larger in the high pressure area behind the shock. At greater heights, as consequence of wave heating which leads to $\bar{p} > p_0$, the tube cross section is expanded beyond the original configuration (cf. Fig. 2).

5.3. Shock formation heights

Figure 7 shows shock formation heights of a large number of wave calculations for different wave energy flux $F_W$ and period $P$ in flux tube model $B$, obtained in cases with and without radiation damping. In addition to the results for longitudinal tube waves (solid), shock heights for acoustic tube waves (dashed) and for ordinary acoustic waves (dash-dot) are shown. All waves are calculated for the same initial atmospheric density and temperature distribution; that is, the density $\varrho_0$ at the bottom of the atmosphere was the same in all calculations. As a general rule, it is seen that the shock heights increase with period for all wave types. This is due to the fact that the non-linear effects, which depend on the wave amplitude and distort the wave profile, take much longer to produce shocks when the wavelength is large.

Let us now compare the results for the three types of waves. In the adiabatic case (Fig. 7), the shock heights for longitudinal tube waves and the acoustic tube waves are almost identical. The acoustic tube wave has, however, systematically greater shock heights because this wave has a slightly larger propagation speed, and thus a larger wave length. In addition, the acoustic tube wave has a smaller initial velocity amplitude for the same $F_W$ because $c_7 < c_7$ [cf. Eq. (62)]. Both effects lead to greater shock heights.

A comparison between the acoustic tube wave and the ordinary acoustic wave shows the influence of the tube geometry. The spreading tube decreases the mechanical flux and with it the wave amplitude, which leads to greater shock heights. Thus, the tube waves have systematically greater shock heights. The spreading of the flux tube is small at low heights, and thus the shock heights for high initial wave fluxes are rather similar for the three types of waves. At greater heights, however, the geometry effect is very pronounced, leading to a great disparity between the shock heights of tube waves and of ordinary waves.

Radiation damping occurs only at low heights. This leads to a considerable decrease of the wave amplitude, and thus as Fig. 7 shows to an increase of the shock heights. As the spreading of the flux tube affects the wave amplitude further, there is an especially large disparity between adiabatic and radiatively damped waves of low initial wave flux. The pronounced damping of short period waves found in earlier work on acoustic waves (Ulmschneider et al., 1977, and USKB 78) is not encountered in our present calculations. In the previous work, the short period waves were primarily affected because of the validity of the diffusion approximation,

$$J - \bar{B} \approx \frac{d^2 \bar{B}}{dt^2},$$

(64)

in Eq. (5). For short period waves the term on the right-hand side of Eq. (64) is quite large. As the optical depth, $\tau$, in the flux tube is about a factor of ten smaller than that of the external medium, the diffusion approximation is no longer valid, and thus radiation damping is less strongly period-dependent. In addition, our tube waves are much less affected by radiation damping than the acoustic waves of the earlier calculation because of the low tube density.

5.4. Tubes of different field strength and spreading

The results discussed above were for flux tube model $B$ where for heights greater than $z_0 = 500$ km only linear spreading was permitted (cf. Fig. 2). In the interior of supergranulation cells, however, where flux tubes are less crowded, individual tubes usually have more room for spreading. In model $A$ we have assumed that exponential spreading continues to $z_0 > 1000$ km (cf. Fig. 2). Here obviously the thin flux tube approximation $B_\phi \approx B_z$ becomes quite bad. Figure 8 shows waves with $P = 30$ s and $F_W = 1.5 \times 10^8$ erg/cm$^2$/s computed in models $A$ and $B$, both displayed at the same time $t = 1567$ s. As the shock formation height is
Fig. 7. Shock formation heights for different types of waves as function of wave period $P$ and initial flux $F_{\nu}$ (indicated in erg cm$^{-2}$ s$^{-1}$). Longitudinal tube waves (solid), acoustic tube waves (dashed) and ordinary acoustic waves (---). Left panel: adiabatic waves. Right panel: radiatively damped waves.

Fig. 8. Wave calculations in flux tube models $A$ (solid) and $B$ (dashed). For comparison both waves with period $P = 30$ s and flux $F_{\nu} = 1.5 \times 10^8$ erg/cm$^2$ s are shown at the same time $t = 1567$ s (cf. Fig. 4).
Fig. 9. Shock formation heights as function of wave period $P$ and initial flux $F_M$ for longitudinal tube waves in different flux tubes of indicated field strength $B_{00}$ at the bottom of the tube. Left panel: adiabatic waves, right panel: radiatively damped waves.

much smaller than the height $z_0$ where models $A$ and $B$ differ mainly the chromospheric behavior is affected by the different rates of spreading. Increased spreading distributes the mechanical energy over a greater cross-sectional area and thus decreases the relative wave amplitude as can be seen from a comparison of the velocities in Fig. 8. Decreased shock strength in turn leads to smaller dissipation and thus to lower mean temperatures $T$ and pressures $\bar{p}$. Once shocks are formed spreading thus critically influences the chromospheric temperature rise and the pressure balance. In addition it is clear from our discussion of shock formation (Sect. 5.3, above) that for $z_0$ much smaller than 500 km shock formation itself could be accelerated (retarded) by a slower (faster) rate of spreading.

So far we have only discussed flux tubes with a fixed magnetic field strength $B_{00} = 1560$ G at the base. To show how a change in the magnetic field strength affects the non-linear behaviour of longitudinal tube waves we have constructed two other tube models. In model $C$, the field strength $B_{00} = 1250$ G has been chosen such that we have $c_A = c_S$ at the bottom of the tube, that is

$$p_{00} = \frac{2}{\gamma + 1} \rho_{00},$$

(65)

while in model $D$ the field strength is increased to $B_{00} = 1670$ G. From Eqs. (3) and (50) it is clear that the shape and the spreading of the flux tube in models $C$ and $D$ is the same as in model $B$. Together with $B_{00}$, the values $p_{00}$ and $\rho_{00}$ are modified. Figure 9 shows the effect on the shock formation heights caused by the variation of field strength $B_{00}$ in the tube for two different wave energies.

In model $C$ and cases of small $B_{00}$, the density $\rho_{00}$ increases in the tube; thus, from Eq. (62), the velocity amplitude $u_0$ and the tube speed $c_T$ decreases. This leads to shock formation at greater heights.

For model $D$ and cases of high field strength $B_{00}$, we have $c_A \rightarrow \infty$, so that the tube becomes more and more rigid; the longitudinal waves thus go over the acoustic tube waves. With increasing $B_{00}$ and thus decreasing density $\rho_{00}$ in the tube, the velocity amplitude $u_0$ of the wave [from Eq. (62)] increases, leading to shock formation at lower height (cf. Fig. 9). On the other hand, decreasing $B_{00}$ leads to greater shock formation heights, less rigidity and thus greater disparity between longitudinal and acoustic tube waves.

6. Conclusions

We have compared longitudinal magneto-hydrodynamic flux tube waves with acoustic tube waves (which propagate in rigid tubes defined by the magnetic field boundary) and with ordinary acoustic waves in plane geometry. For the case of isothermal gravitationally-stratified atmospheres, phase relations are derived for the linear theory (Appendix A). Longitudinal tube waves
become acoustic tube waves when the distensibility, that is, the
time dependent breathing of the flux tube is neglected. Acoustic
tube waves in turn reduce to ordinary acoustic waves when the
tube geometry, i.e. the spreading of the tube is neglected.

The time-dependent non-linear longitudinal tube wave equa-
tions were solved with the effect of radiation damping taken into
account. We have derived Hugoniot relations valid for shock wave
propagation in magnetic flux tubes which in the limit of vanishing
distensibility go over to the ordinary Hugoniot relations.

The linear and non-linear computations both show that in the
chromosphere longitudinal tube waves relative to ordinary acous-
tic waves have much reduced wave amplitudes (Figs. 3–6 and 8).
This is due to the fact that the wave energy is spread over a with
height increasing cross-sectional area. Decreasing shock ampli-
tudes lead to smaller mechanical dissipation, but also to dimin-
ished radiation loss. Thus the time averaged chromospheric
temperatures and pressures in the flux tube are reduced relative to
the ordinary acoustic case of the same input flux. For longitudinal
tube waves we find in addition that chromospheric heating
expands the flux tube relative to its initial radiative equilibrium
configuration (Fig. 2).

Due to the spreading of the tubes both longitudinal and
acoustic tube waves form shocks at greater height than ordinary
acoustic waves do (Fig. 7). This is caused by the small non-linear
wave distortion resulting from the slow growth of the wave
amplitude. The disparity in shock formation height is greatest for
long period waves where the amount of spreading per wavelength
is largest. The delayed shock formation and dissipation in
longitudinal waves allows more wave energy to reach great heights
where the density and consequently the emission is small. Thus
the chromospheric temperature gradient steepens much more with
increasing wave period than is the case for ordinary acoustic waves
(Figs. 3–6).

Our grey LTE optically-thin method of treatment of radiation
damping is more general than methods which use Newtons law
of cooling and adopt constant values of the radiative relaxation time
(see Appendix B). Yet even this improved radiation treatment is
highly simplistic in chromospheric situations. Test calculations
(Schmitz et al., 1985) have shown that the low density in flux tubes
leads to strong non-LTE effects in the dominant $H^\text{+}$ emitter.
Similarly effects of resonance scattering and strong non-LTE
effects are expected for the dominant Mg II and Ca II line emission
of the middle chromosphere.

Thus a quantitative discussion of the chromospheric tempe-
rament and density stratification, as well as heating and cooling rates
must await future investigations. Nevertheless the qualitative
computations of our present work show that longitudinal tube
waves readily transport mechanical energy to chromospheric
heights and can dissipate it there through shocks. As the wave
energy generation in flux tubes is much more efficient
(Ulmschneider and Stein, 1982) and as the tube spreading allows the
energy to be transported to great heights there seems to be a good case
for longitudinal tube waves as a chromospheric heating mecha-
nism.

Finally for solar type flux tubes it was found that longitudinal
and acoustic tube waves are very similar (Fig. 7). The effect of
the distensibility thus appears much less important than the effect
of the tube spreading. If the magnetic field strength in the tube is
increased longitudinal waves become identical to acoustic tube
waves. Decreasing the field strength in the tube leads to increasing
influence of the distensibility and acts to magnify the disparity
between longitudinal and acoustic tube waves (Fig. 9).

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**Appendix**

**A. Polarization factors in the linear theory**

The solution of the longitudinal tube wave Eqs. (1) to (4) for
linearized cases assuming adiabatic wave propagation in an
isothermal atmosphere with temperature $T_0$ were first given by
Defouw (1976) and subsequently generalized by Roberts and
Webb (1978). For the density-, pressure-, velocity-, magnetic field-, and
cross-sectional perturbations the solutions are written, $\psi$ being an
arbitrary amplitude

$$
\psi' = \psi_{00} \psi \mathbf{e}^{-\frac{3}{4} H \mathbf{e}^{-(\text{ot} - kr)}} ,
$$
(A.1)

$$
p' = p_{00} \psi \mathbf{e}^{-\frac{3}{4} H \mathbf{e}^{-(\text{ot} - kr)}} ,
$$
(A.2)

$$
T' = T_{00} \psi \mathbf{e}^{\frac{3}{2} H \mathbf{e}^{-(\text{ot} - kr)}} ,
$$
(A.3)

$$
u' = \psi_{00} \mathbf{e}^{\frac{3}{2} H \mathbf{e}^{-(\text{ot} - kr)}} ,
$$
(A.4)

$$
B' = B_{00} \psi \mathbf{e}^{\frac{3}{2} H \mathbf{e}^{-(\text{ot} - kr)}} ,
$$
(A.5)

$$
A' = A_{00} \psi \mathbf{e}^{\frac{3}{2} H \mathbf{e}^{-(\text{ot} - kr)}} ,
$$
(A.6)

where $H$ is the scale height and $\psi_{00}, p_{00}, T_{00}, A_{00}$ are values of the
undisturbed atmosphere at the bottom of the flux tube. The values
$\tilde{\psi}, \tilde{p}, \tilde{T}, \tilde{u}, \tilde{B},$ and $\tilde{A}$ are polarization factors which following Hines
(1960) can be written as non-trivial solutions of the characteristic
equation

$$
\tilde{\psi} = -i \left[ \frac{\gamma - 1}{\gamma H} \frac{c_s^2}{c_A^2} + \frac{1}{4 H} \right] + k ,
$$
(A.7)

$$
\tilde{p} = \frac{3 \gamma - 4}{4 H} + \gamma k ,
$$
(A.8)

$$
\tilde{T} = \tilde{\psi} \mathbf{e}^{\frac{3}{2} H \mathbf{e}^{-(\text{ot} - kr)}} \left( \frac{3}{4} + \frac{c_s^2}{c_A^2} \right) + (\gamma - 1) k ,
$$
(A.9)

$$
\tilde{u} = \tilde{\psi} \mathbf{e}^{\frac{3}{2} H \mathbf{e}^{-(\text{ot} - kr)}} ,
$$
(A.10)

$$
\tilde{B} = -\tilde{A} = -\frac{c_s^2}{c_A^2} \tilde{\psi} ,
$$
(A.11)

where $\gamma$ is the ratio of specific heats, $c_s$ is the sound velocity and $c_A$
the Alfvén velocity.

For the acoustic tube waves the solutions and polarization
factors may be obtained from the limit $c_A \to \infty$. For cases with
no gravity ($H \to \infty$), the polarization factors go over to those of
Roberts and Webb [1978, Eq. (48), note the different sign of the
propagation vector $k$]. For $\omega \to \infty$ one has $k \to \infty$ and the
perturbations $u, \psi, p', A', T'$ are in phase, while $B'$ is $180^\circ$ out of
phase. For $\omega \to \omega_c$, where $\omega_c$ is the cut-off frequency, one has $k \to 0$ and
the perturbations $T', p', A'$ lead $u$ by $90^\circ$ in phase, while $u$ leads
$\psi$ and $B'$ by $90^\circ$ in phase. This situation is similar to that for
the ordinary acoustic waves (Hines, 1960) except that for $\omega \to \omega_c$ the
phase of $p'$ is different by $180^\circ$. Phase and group velocities are
similar to those of the ordinary acoustic wave case (Hines, 1960)
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for purely vertical propagation. Like in the acoustic wave case the cut-off frequency \( \omega_c \) is explained as the fundamental resonance of the entire tube, as is seen from the fact that

\[
v_{\text{ph}} \to \infty \quad \text{for} \quad \omega \to \omega_c.
\]

B. Radiation treatments using Newton’s law of cooling

The radiation treatments of Webb and Roberts (1980) and of Roberts (1983) use Newton’s law of cooling

\[
\frac{dQ}{dt}_{\text{rad}} = -\frac{c_v}{\tau_R} (T_e - T)
\]

and assume a constant radiative relaxation time \( \tau_R \). Here \( \frac{dQ}{dt}_{\text{rad}} \) is the net radiative cooling rate (erg g\(^{-1}\) s\(^{-1}\)) and \( c_v \) is the specific heat at constant volume.

Using Eqs. (5), (8), and (9) we have a more general law

\[
\frac{dQ}{dt}_{\text{rad}} = \frac{c_v}{\tau_R} \frac{dT}{dt} \bigg|_{\text{rad}} = -4\kappa \sigma (T_e^4 - T^4)
\]

The radiative relaxation time

\[
\tau_R = \frac{c_v}{16\kappa \sigma T^3}
\]

varies rapidly with height (cf. Ulmschneider 1971, Eq. (16) and Table 2). Assuming \( T_e \approx T \) it is seen by expanding (B.2) that Newton’s law of cooling is a special case of the more general formula (B.2).

References