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Acoustic waves in the solar atmosphere

VIII. Extrapolation of the solution in time

P. Ulmschneider

Institut für Theoretische Astrophysik, Im Neuenheimer Feld 561, D-6900 Heidelberg, Federal Republic of Germany

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Summary. An accurate method for prediction of the physical state in a time-dependent acoustic wave calculation is given. The method extrapolates the wave in time along the C^+ characteristics and is useful for a transmitting boundary condition.

Key words: hydrodynamics – numerical methods – shock waves – stars: chromospheres – sun: chromosphere

1. Introduction

In time-dependent acoustic wave calculations where the fully dynamic coupling between the hydrodynamics, the non-equilibrium thermodynamics and the radiation field is consistently treated, implicit iteration schemes are usually employed to determine the state at the new time level from the known state at the old time level. For these iterations a highly accurate initial estimate of the state at the new time level is helpful to increase the speed of the overall convergence.

In addition the limited size of computer memories necessitates the introduction of computational boundaries where waves and flows can be transmitted out of the computational domain with little reflection. In the Lagrangian scheme used in our acoustic wave computations two characteristics are available to determine the three unknown physical variables (two thermodynamic variables and the flow velocity). Transmission at the boundary is accomplished by accurately extrapolating from the computational domain the third physical variable. Such a transmitting boundary condition is not rigorous in a physical sense and remains an unsatisfactory computational limitation because it neglects the information entering from outside into the computational domain.

Under the names open or radiating boundaries the transmitting boundaries are well known in astrophysical and geophysical fluid dynamics. Leibacher (1971), Stein and Schwartz (1972), Klein, Stein and Kalkofen (1976), Orlanski (1976), Gustafsson and Sundstöm (1978), Oliger and Sundstöm (1978), Reynolds (1978), Engquist and Majda (1979), Israeli and Orszag (1981) as well as Strikwerda (1981) considered waves propagating in $+x$-direction and satisfying the condition

$$\left(\frac{1}{c}\frac{\partial}{\partial t}-\frac{\partial}{\partial x}\right)u=0, \tag{1}$$

where t is time, x is Eulerian distance and c is the phase speed. In some of this work transmitting boundaries were achieved by fitting analytic wave calculations for isothermal gravitational atmospheres to the boundary points. More elaborate methods use the actual phase velocity of the non-linear waves and estimate the physical variables at the transmitting boundary similar to those for simple waves.

For large amplitude acoustic waves an exact transmitting boundary condition is possible for simple waves (Landau and Lifshitz, 1959; p. 366ff, Courant and Friedrichs, 1948, pp. 59f, 87f, 92ff, see also Hammer and Ulmschneider, 1978) in homogenous atmospheres because the information entering via the C^- characteristic from outside the computational domain is known. This transmitting boundary condition (which specifies e.g. that the flow velocity $u=$ const. along the C^+ characteristics emanating from the boundary) has been successfully used by Bohn (1974) and Ulmschneider et al. (1977) even for gravitational atmospheres.

In the present work an extrapolation method is suggested which essentially follows the slowly varying evolution of individual phases of the wave. It is shown that this method is more accurate than the previous simple wave extrapolation method used in our stellar atmospheric wave calculations. In Sect. 2 the extrapolation method and its accuracy is described. Section 3 discusses the use of the extrapolation method for a transmitting boundary condition. Conclusions are given in Sect. 4.

2. Method of extrapolation and accuracy of the procedure

We assume (cf. Fig. 1) that the physical variables are known at N Lagrangian height points $a(i)$, $i=1, N$ at a *current* time level $t_j=t$ and at the *old* time level $t_{j-1}=t-\Delta t$. We further assume that at the two time levels the physical variables are known at NSH shocks at positions ash (n,t_j), ash (n,t_{j-1}). Here the index $n=1$, NSH identifies the shocks. Knowledge of the physical variables at grid points (shown as dots in Fig. 1) and shock points (shown as squares in Fig. 1) is desired at the *new* time level $t_{j+1}=t+\Delta t$. From every grid point (except the lowest one) and from every shock point of the current time level we follow the C^+ characteristics and the shock paths back to the old time level. For this purpose the slopes of the characteristics are given by (Ulmschneider et al., 1977)

$$y=\frac{da}{dt}=\frac{c^4}{c_0^3}\exp\left(\frac{(S-S_0)\mu}{R}\right), \tag{2}$$

where c is the sound velocity, S the entropy, μ the mean molecular weight and R the universal gas constant. Subscript 0 indicates initial values at the same Lagrange height. The intersection points

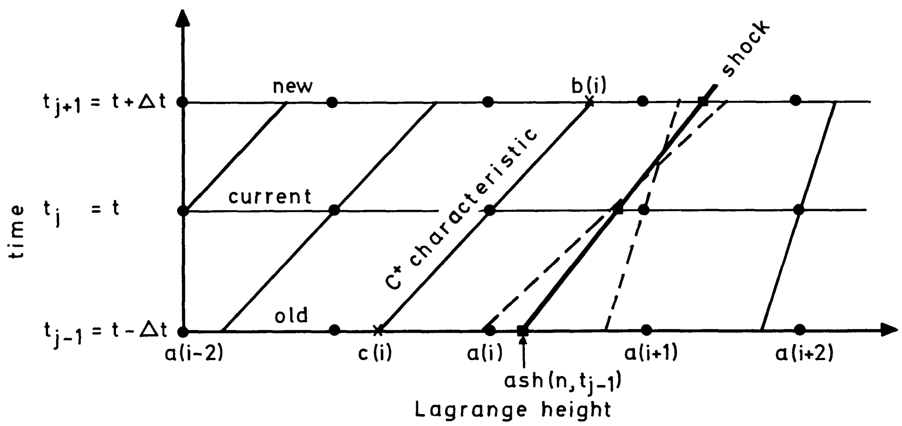

Fig. 1. C^+ characteristics and the shock path in the time-height plane intersecting the three time levels $t_{j-1} = t - \Delta t$ (old), $t_j = t$ (current) and $t_{j+1} = t + \Delta t$ (new). The solution is presumed known at the old and current time levels at Lagrangian grid points $a(i)$ and shock points ash. Grid points are indicated by dots and shock points by squares

$c(i)$ of the C^+ characteristics with the old time level (cf. Fig. 1) are computed iteratively and the physical variables (including values for y) at these points are determined by interpolation. Note that in computation schemes using the method of characteristics these intersection points and interpolated physical variables are computed anway and thus need only to be saved.

The physical variables at the new time level are now calculated by linear extrapolation over time (cf. Fig. 1). Using the extrapolated knowledge of y and of the shock speed, the intersection points of the C^+ characteristics, $b(i)$, and of the shock paths, ash (n, t_{j+1}), with the new time level are determined iteratively. C^+ characteristics immediately behind the shocks can overtake the shock. Likewise, shocks can overtake the C^+ characteristics immediately in front of the shock. Characteristics of this type (shown dashed in Fig. 1) are not used in the extrapolation procedure. As the set of physical variables at the new time level is not desired at height points $b(i)$, but rather at the heights $a(i)$, the extrapolated values finally are interpolated back onto the standard grid $a(i)$. An estimate of the physical variables at the lowermost gridpoint $a(1)$ cannot be obtained with such a procedure. Here linear extrapolation along the time axis gives a low accuracy result.

For a series of time steps in a solar atmospheric wave calculation similar to that of Ulmschneider et al. (1978) the differences between the extrapolated and the computed values for three typical physical variables, the velocity u, the temperature perturbation ΔT and the radiative damping function D were compared, The maximum errors ϵ_U, ϵ_T, and ϵ_D at any grid or shock point found in the entire series of wave calculations are listed in Table 1 (entry with $w = 0$). The velocity error is defined as

$$\epsilon_U = \mathrm{Abs}((u_{\mathrm{EXT}} - u)/u_{\mathrm{MAX}}), \tag{3}$$

where u_{EXT} is the extrapolated velocity, u is the computed velocity and u_{MAX} is the maximum absolute velocity in the wave calculation. ϵ_T and ϵ_D are defined similarly. It is seen that the velocity can be extrapolated to better than 2% over the entire height slab. A similar accuracy is found for the extrapolation of the temperature perturbation. This favourable result can be attributed to the smooth and monotonic behaviour of both velocity and temperature along the C^+ characteristics which follow the individual phases of the wave.

In addition Table 1 shows the accuracy of the extrapolation for the radiative damping function D (which is defined e.g. in Eq. (9) of Ulmschneider et al., 1978). Due to the steep rise of the shock strength and the rapid decrease of the opacity with height, this function is strongly peaked in the middle of the atmosphere and in addition exhibits rapidly decreasing cooling zones behind the shocks. In spite of the considerably increased variability of this function it is seen that the accuracy of the extrapolation is better than about 5%. For all variables the largest errors occur in the steep regions behind the topmost shocks. Here errors in the position of the C^+ characteristics lead to extrapolation errors. Relatively large errors occur also at the lowermost shocks. They are due to the fact that these shocks have not yet fully developed their sawtooth shape and thus their profile-evolution is difficult to predict.

Table 1. Maximum error of the proposed extrapolation method for a series of time steps in an acoustic wave calculation. ϵ_U is the error in the velocity after Eq. (3), ϵ_T is the error of the temperature perturbation, and ϵ_D that of the radiative damping function. The parameter w specifies different weights for parabolic extrapolation. The values in brackets are for the case where the physical quantities are extrapolated using the simple wave assumption.

w	ϵ_U	ϵ_T	ϵ_D
0	1.9E−2	1.9E−2	5.3E−2
0.2	1.9E−2	1.6E−2	6.0E−2
0.5	1.7E−2	1.6E−2	6.9E−2
0.9	2.9E−2	2.9E−2	8.1E−2
(0)	(4.3E−2)	(8.3E−2)	(1.2E−1)

It is possible to extend the C^+ characteristics backwards by yet another time level and use three known time steps for a parabolic extrapolation. Parabolic and linear extrapolations can be weighted using the equation.

$$u_{\mathrm{EXT}} = u_{\mathrm{PARAB}} w + u_{\mathrm{LINEAR}}(1 - w), \tag{4}$$

where $0 \leq w \leq 1$ is an extrapolation weight. Parabolic extrapolation does not improve the accuracy of the result as is seen from Table 1.

Table 1 also shows the case where the simple wave assumption (cf. Ulmschneider et al., 1977) was made in the extrapolation procedure. In this assumption the physical variables do not get extrapolated but remain constant along the C^+ characteristics.

It is seen that the simple wave assumption leads to errors which are larger by more than a factor of two compared with our present method.

3. Transmitting boundary conditions

Extrapolations are often used to determine physical variables at the boundary points to generate a transmitting boundary condition. For this it is sufficient to extrapolate one variable e.g. the velocity u (cf. Ulmschneider et al., 1977) because their is enough information along the two remaining characteristics to compute the other two unkown variables. As interior points (where all three characteristics are available) can be considered as perfectly transmitting points it is clear that the transmission quality at a boundary depends on how well the one specified variable can be predicted. The simple wave assumption which was employed by Bohn (1974) and by Ulmschneider et al. (1977) as well as by Muchmore and Ulmschneider (1985) improved the transmission quality compared to older methods. Our present method improves on this by removing the homogeneous atmosphere assumption inherent in the simple wave picture. A note of caution is in order here. When transmitting boundary conditions are used, then the error made by neglecting the incoming information along the C^- characteristics must be carefully considered.

4. Conclusions

It has been shown that linear extrapolation along the C^+ characteristics is a powerful method which allows the prediction of the physical variables to an accuracy of a few percent. This is possible because the C^+ characteristics are lines which follow the individual phases of the wave and thus take into account the distortion of the wave profile caused by the nonlinear terms in the hydrodynamic equations which are important in waves with large amplitudes. Along these characteristic lines the variation of the physical variables introduced by the density stratification, the radiation damping and the shock dissipation is particularly smooth. Parabolic extrapolation along the C^+ characteristics did not improve the accuracy. Compared with the simple wave assumption which was used as transmitting boundary condition in previous work of this series the present method is at least a factor of two more accurate. The present method automatically gives exact extrapolations when homogeneous atmospheres (simple wave assumption) or small amplitude sound waves (displacement of the wave profile with the sound speed) are considered. The method is well suited for use as a transmitting boundary condition.

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