The Role of Fine-Scale Magnetic Fields on the Structure of the Solar Atmosphere
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ADIABATIC LONGITUDINAL–TRANSVERSE MAGNETOHYDRODYNAMIC TUBE WAVES

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Abstract. We compute the propagation of adiabatic magnetohydrodynamic waves along thin flux tubes in the solar atmosphere. The time-dependent development and amplitude growth of the waves, excited at the foot of the photosphere as pure transverse waves, was followed into the lower chromosphere. Strong mode-coupling to longitudinal waves was found. The swaying of the tube which increases with height resulted in a lifting of the entire tube mass which we attribute to centrifugal forces. This lifting resulted in adiabatic cooling of the tube.

1. Introduction
As discussed by Spruit (1982) thin magnetic flux tubes allow three types of tube wave modes: longitudinal, transverse and torsional waves. Despite the fact that these waves are probably all excited very efficiently it is interesting to investigate their non-linear coupling especially as this may have great importance for both chromospheric and coronal heating (cf. Ulmschneider 1987). Hollweg et al. (1982) have computed the time-development of coupled longitudinal and torsional waves. They showed that from a purely torsional excitation longitudinal waves of considerable energy developed. In the present paper we want to report similar time-dependent work on coupled longitudinal and transverse waves.

2. Method
Following Spruit (1981) we assume a vertically oriented thin magnetic flux tube (cf. Fig. 1) in which a mass element of width da at time t=0 can be uniquely identified by the Lagrange height, a, measured in the outward direction from the level where the continuum optical depth outside the tube is one. At a=0 the tube has a diameter of 100 km and a magnetic field strength of \( B_0 = 1500 \) G (cf. Tab. 1). The tube is assumed to spread with height almost exponentially in pressure balance in an external unmagnetized radiative equilibrium atmosphere similarly as

![Fig. 1 Flux tube geometry](image-url)
discussed by Herbold et al. (1985). At some later time \( t \) (cf. Fig. 1) the tube is assumed to have kinks and the cross-sectional area \( A \) may be changed. The mass element has now the width \( dl \) and is displaced to a position described by the arc length \( l(a,t) \). The unit vector in direction \( l \) is defined by:

\[
\hat{\mathbf{l}} = \frac{\partial l}{\partial t} = \frac{1}{a} \left( \frac{\partial l}{\partial a} \right)_a \mathbf{t} = (1_x, 1_y, 1_z), \quad \text{with} \quad 1_x^2 + 1_y^2 + 1_z^2 = 1, \quad (1)
\]

and \( l_a \) defined below. A curvature vector is defined by

\[
\mathbf{p} = \frac{\partial \hat{\mathbf{l}}}{\partial t} = \frac{1}{a} \left( \frac{\partial \hat{\mathbf{l}}}{\partial a} \right)_a \mathbf{t} = \hat{\mathbf{l}}. \quad (2)
\]

The position of the mass element may also be described by the radius vector \( \mathbf{r} \) from an Eulerian coordinate system (cf. Fig. 1). The magnetic field is assumed to be exclusively in the \( \hat{\mathbf{l}} \) direction. The conservation of mass requires

\[
\rho(a,t) A(a,t) \, dl = \rho_0(a,t) A_0(a,t) \, da, \quad (3)
\]

where \( \rho \) is the density and the subscript \( o \) denotes values at time \( t=0 \). Eq. (3) defines a scale factor

\[
l_a = \frac{\partial l}{\partial a} = \frac{\rho_0 A_0}{\rho A}. \quad (4)
\]

We first consider Euler's equation:

\[
\rho \left( \frac{\partial \mathbf{v}}{\partial t} + \mathbf{v} \cdot \nabla \mathbf{v} \right) = -\nabla p - \frac{1}{4\pi} \mathbf{B} \times (\nabla \times \mathbf{B}) + \rho \mathbf{g}. \quad (5)
\]

For the transverse component of this Eq. one has (Spruit 1981):

\[
\rho \left( \frac{\partial \mathbf{v}}{\partial t} + \mathbf{v} \cdot \nabla \mathbf{v} \right)_a - \left[ \hat{\mathbf{l}} \left( \frac{\partial \mathbf{v}}{\partial t} \right)_a \right] \hat{\mathbf{l}} = - \left[ \hat{\mathbf{l}} \times (\nabla (p + \frac{B^2}{2\mu_0})) \right] \times \hat{\mathbf{l}} + \frac{B^2}{4\pi} \mathbf{K} + \rho (l_x \mathbf{g})_x \hat{\mathbf{l}} \quad (6)
\]

As the transverse wave moves also mass outside the tube we replace \( \rho \) in the inertia term by \( \rho + \rho_e \). Using \( p + B^2/\mu_0 = p_e(z) \) one finds

\[
\left[ \frac{\partial \mathbf{v}}{\partial t} \right]_a - \left[ \hat{\mathbf{l}} \left( \frac{\partial \mathbf{v}}{\partial t} \right)_a \right] \hat{\mathbf{l}} = \frac{\rho c_A^2}{\rho + \rho_e} \left[ \hat{\mathbf{l}} \left( \frac{\partial \mathbf{v}}{\partial a} \right)_a \right] + \frac{\rho - \rho_e}{\rho + \rho_e} (\mathbf{g} + gl_z \hat{\mathbf{l}}) \quad (7)
\]

with \( c_A \) = Alfvenspeed. For the longitudinal component one has:

\[
\rho \hat{\mathbf{l}} \cdot \left( \frac{\partial \mathbf{v}}{\partial t} \right)_a = - \frac{1}{a} \left( \frac{\partial \mathbf{v}}{\partial a} \right)_a \hat{\mathbf{l}}, \quad -\rho g l_z. \quad (8)
\]

We now consider the induction and continuity equations:

\[
\frac{\partial \mathbf{B}}{\partial t} = \nabla \times (\mathbf{v} \times \mathbf{B}) = \nabla (\mathbf{v} \cdot \mathbf{B}) + (\mathbf{B} \cdot \nabla) \mathbf{v} - \nabla (\mathbf{v} \cdot \mathbf{B}) - (\nabla \cdot \mathbf{v}) \mathbf{B} \quad (9)
\]

\[
\frac{\partial \mathbf{v}}{\partial t} + (\mathbf{v} \cdot \nabla) \rho + \rho (\mathbf{v} \cdot \nabla) \mathbf{v} = 0 \quad (10)
\]

Using these Eqs. we find for the transverse component:
\[
\left[ \frac{\partial}{\partial t} \right]_{a} = -\frac{1}{\tau_{a}} \left[ \frac{\partial}{\partial x} \right]_{a} - \left[ \frac{\partial}{\partial t} \right]_{a} \] 
\text{and for the longitudinal component:}
\[
\left[ \frac{\partial}{\partial t} \right]_{a} = \frac{\rho}{\rho_{e}} \left[ \frac{\partial}{\partial x} \right]_{a} + \frac{B}{\tau_{a}} \hat{\gamma} \left[ \frac{\partial}{\partial t} \right]_{a} 
\] 
(12)
From the time derivative of \(B^2/8\pi = p_e - p\) we have
\[
\frac{\rho c_A^2}{B} \left[ \frac{\partial}{\partial t} \right]_{a} = \left[ \frac{\partial}{\partial x} \right]_{a} \left( p_e - p \right)_{a} = v_{z} \frac{dp_e}{dz} - \left[ \frac{dp}{\partial t} \right]_{a}, 
\text{and with Eq. (12):}
\hat{\gamma} \left[ \frac{\partial}{\partial t} \right]_{a} = -\frac{1}{\rho c_A^2} \left[ \frac{\partial}{\partial x} \right]_{a} - \frac{1}{\rho \tau_{a}} \frac{\partial}{\partial x} - \frac{v_{z}}{\tau_{a}} \frac{dp_e}{dz} 
\] 
(13)
Eliminating \(\rho\) and \(p\) in favour of the sound speed \(c\) and the entropy \(S\) we now have the following 5 equations, Euler's equation:
\[
\hat{\gamma} \left[ \frac{\partial}{\partial t} \right]_{a} + \frac{1}{\tau_{a}} \left[ \frac{2c}{T} \right] - \frac{c_A^2}{\tau_{a} \left[ \frac{\partial}{\partial x} \right]_{a}} - \frac{2}{\gamma - 1} \left[ \frac{\partial}{\partial x} \right]_{a} \frac{\gamma}{\tau_{a}} \frac{dp_e}{dz} + g_1 \frac{dp_e}{dz} = 0 
\] 
(14)
\[
\hat{\gamma} \left[ \frac{\partial}{\partial t} \right]_{a} - \frac{c_A^2}{\tau_{a} \left[ \frac{\partial}{\partial x} \right]_{a}} - \frac{\rho}{\rho + \rho_e} \left[ \frac{\partial}{\partial x} \right]_{a} - \frac{\rho - \rho_e}{\rho + \rho_e} \left( g + g_1 \right) = 0 
\] 
(15)
Induction and continuity equations:
\[
\hat{\gamma} \left[ \frac{\partial}{\partial t} \right]_{a} + \frac{2c}{T} \left[ \frac{\partial}{\partial x} \right]_{a} - \frac{c_A^2}{\tau_{a} \left[ \frac{\partial}{\partial x} \right]_{a}} - \frac{\gamma}{\tau_{a}} \frac{dp_e}{dz} = 0 
\] 
(16)
\[
\hat{\gamma} \left[ \frac{\partial}{\partial t} \right]_{a} - \frac{c_A^2}{\tau_{a} \left[ \frac{\partial}{\partial x} \right]_{a}} - \frac{\rho}{\rho + \rho_e} \left[ \frac{\partial}{\partial x} \right]_{a} = 0 
\] 
(17)
where \(c_T = \left[ c^2 c_A^2 / (c^2 + c_A^2) \right]^{1/2}\) is the tube speed.

Entropy conservation equation:
\[
\left[ \frac{\partial}{\partial t} \right]_{a} = \frac{\partial}{\partial x} \frac{\partial S}{\partial x} \] 
rad 
(18)
Going over to characteristic form these equations read:
\[
\hat{\gamma} \left[ \frac{\partial}{\partial t} \right]_{a} + \frac{2c}{c_T} \left[ \frac{\partial}{\partial x} \right]_{a} + \frac{\partial}{\partial x} \frac{\partial S}{\partial x} \frac{\partial}{\partial t} = \frac{\partial}{\partial x} \left[ \frac{\partial S}{\partial x} \right]_{a} \] rad 
\frac{\partial}{\partial x} \frac{\partial S}{\partial x} \frac{\partial}{\partial t} = 0 
(19)
along the \(C_{2x}^{0}\) characteristics given by \(\frac{\partial a}{\partial t} = \pm \frac{c_T}{1} \frac{\partial}{\partial x} \) 
(20)
where Eqs. (19), (20) are those of Herbold et al. (1985) for \(l_{z} = 1\), and
\[
\left( 1 - l_{x}^2 \right) \frac{\partial v_{x}}{\partial t} - \left( 1 - l_{y}^2 \right) \frac{\partial v_{y}}{\partial t} = c_f \frac{\partial l_{x}}{\partial t} - \frac{\partial}{\partial x} \frac{\partial S}{\partial x} \frac{\partial}{\partial t} = 0 
\] 
(21)
\[
\left( 1 - l_{y}^2 \right) \frac{\partial v_{y}}{\partial t} - \left( 1 - l_{x}^2 \right) \frac{\partial v_{x}}{\partial t} = c_f \frac{\partial l_{y}}{\partial t} - \frac{\partial}{\partial x} \frac{\partial S}{\partial x} \frac{\partial}{\partial t} = 0 
\] 
(22)
along the \(C_{1}^{0}\) characteristics given by \(\frac{\partial a}{\partial t} = \pm \frac{c_f}{1} \frac{\partial}{\partial x} \equiv \pm \sqrt{\frac{\rho + \rho_{e}}{\rho \rho_{e}}} \frac{c_A}{1} \) 
(23)

3. Results
We have solved the time-dependent equations described above for a case of an adiabatic wave and purely transverse shaking at the bottom of the tube assuming
\[
v_{\perp} = -v_{o} \sin(2\pi t/P), \quad v_{\perp} = 0, \quad v_{\parallel} = 0, 
\] 
(24)
with \(v_{o} = 0.5 \text{ km/s}\) and transmitting boundary conditions at the top.
Transmission was achieved assuming \(v_{\parallel} = \text{const} \) along \(C_{2x}^{0}\) and \(v_{\perp} = v_{\perp} \)
const along C. The wave period was taken to be P=45s. As this shaking took place only in the x-direction the wave is confined to the x-z plane.

Fig. 2 shows a snapshot of the wave after 755 time steps. Note that the physical variables are shown here as function of the Lagrange height a. The curve labeled x shows the horizontal position of the center of

The amplitude of \( v_x \) increases as a function of height due to flux conservation. With the propagation speed \( c \) roughly constant (Tab. 1) the amplitude of \( v_x \) grows roughly like \( r^{-1/2} \). It is seen that \( v_x \) has a maximum of 4 km/s at 800 km height. At that height the x-curve has an inflection point where the gas elements in the tube, due to the motion of the wave in +z direction, are horizontally displaced with maximum speed in +x direction.

Fig. 2 shows that in addition to the transverse wave a longitudinal wave is generated where the vertical velocity \( v_z \) due to mode coupling shows a surprisingly large amplitude of 2 km/s, that is roughly one half the amplitude of \( v_x \). Moreover it is seen in Fig. 2 that the longitudinal wave has twice the frequency of the transverse wave. This is understood if one decomposes the curvature force vector into vertical and horizontal components. In one wavelength of the x-curve the z-component of the curvature force vector changes sign four times while the x-component changes sign only two times.

Another surprising result of the present calculation is that the entire mass of the tube appears to be lifted relative to the initial position. Fig. 2 shows as function of height the distances z-a which are the height differences between the current (Eulerian) position of the
gas elements and the (Lagrange) height (= the height at the start of the
computation). It is seen that the bottom of the tube is lifted by about
40 km, the top by about 90 km, while everywhere else the tube is lifted
by at least 20 km. The reason for this lifting appears to be the
increase of the horizontal motion with height. As the bottom of the tube
does not move much in the horizontal direction, the large horizontal
motion of the higher tube regions generates centrifugal forces, which,
always outwardly directed, lead to the lifting of the tube. The result
of this lifting is an adiabatic cooling of the entire tube as can be
seen in Fig. 2 by comparing the current T and initial $T_0$
temperatures.

4. Conclusions
We found that purely transverse excitation of magnetic flux tubes will
lead to transverse waves with amplitudes strongly growing with height.
In addition, due to strong mode coupling, longitudinal waves were
generated with amplitudes only a factor of two smaller than those of the
transverse waves. The wave period of the longitudinal wave was one half
that of the transverse wave. The horizontal swaying of the tube
increases strongly with height and resulted in centrifugal forces which
lead to a lifting and adiabatic cooling of the entire gas column in the
flux tube. This part of our calculation would certainly be modified if
radiative damping were taken into account. In addition, the analogy to
other free-boundary organ pipe-type wave motions suggests strong
resonance effects i.e. dependences on the chosen wave period. These
effects will be considered in our forthcoming work elsewhere.

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