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ON THE GENERATION OF FLUX TUBE WAVES IN STELLAR CONVECTION ZONES. I. LONGITUDINAL TUBE WAVES DRIVEN BY EXTERNAL TURBULENCE

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ABSTRACT

We derive the source functions and the energy fluxes for wave generation in magnetic flux tubes embedded in an otherwise magnetic field-free, turbulent, and compressible fluid. The calculations presented here assume that the tube interior is not itself turbulent, e.g., that motions within the flux tube are due simply to external excitation. Specific results for the generation of longitudinal tube waves are presented.

Subject headings: hydromagnetics — stars: interiors — stars: late-type — turbulence — wave motions

I. INTRODUCTION

Wave generation by turbulent motions in the outer convection zones of stars has long been thought to be central to the heating of stellar chromospheres and coronae (e.g., Biermann 1946; Schwarzschild 1948). The early suggestions were followed by a number of detailed studies in which the generation of acoustic waves (Lighthill 1952; Proudman 1952; Stein 1967; Renzini *et al.* 1982; Bohn 1980, 1984) as well as MHD waves (Kulsrud 1955; Osterbrock 1961; Parker 1964; Kuperus 1965; Stein 1981; Ulmschneider and Stein 1982; Musielak and Rosner 1987, 1988) was considered. These latter calculations are all based on the assumption of a uniform and weak background magnetic field, an assumption which is contradicted by solar and stellar observational evidence for inhomogeneous and locally strong magnetic fields (see Harvey 1977; Stenflo 1978; Robinson, Worden, and Harvey 1980); thus, at least the solar magnetic field has instead a "flux tube" structure, and flux tube waves carrying the wave energy away from the convection zone may well be responsible for heating at least some portion of the outer atmospheric layers (see Spruit and Roberts 1983). Later, more refined, calculations of wave generation in stellar convection zones confirmed on quantitative grounds that both purely acoustic waves (Ulmschneider and Bohn 1981; Bohn 1984) and MHD waves for an assumed homogeneous magnetic field (Musielak and Rosner 1987, 1988) are insufficient to explain the UV and soft X-ray fluxes observed by the *IUE* and *Einstein Observatories* (Linsky 1981; Vaiana *et al.* 1981; Rosner, Golub, and Vaiana 1985). Indeed, the calculations of Musielak and Rosner (1988) showed that the MHD energy fluxes for homogeneous background magnetic fields are even *less* than those obtained for acoustic waves; in any case, quite aside from the question of matching the required (observed) absolute level of total energy supply, the associated energy fluxes do not vary enough for a given spectral type in order to explain the variation in the observed UV and X-ray fluxes. For these various reasons, it behooves us to determine the likely wave energy flux associated with waves propagating in an inhomogeneous, magnetized background. The work reported in this paper represents an initial effort in this direction.

Recently, Goldreich and Kumar (1988) have discussed the difference between "free" and "forced" turbulence in the efficiency of generating and damping sound waves. They demonstrate that the difference in generation efficiency between these two cases can be very large, and that the coupling of gravity to entropy fluctuations leads to a fluctuating buoyancy force which is responsible for maintaining the turbulence and which leads to dipole emission of sound waves. This point was not appreciated in earlier discussions of this problem (Goldreich 1987), in which the term "dipole emission" was used instead to describe corrections to quadrupole emission resulting from the anisotropy of the background stratified medium (see Stein 1967; Bohn 1984). Goldreich and Kumar further showed that the effective damping rate of sound waves by forced turbulence is very likely to be large and will completely dominate the corresponding scattering processes. This damping process may become important in the evaluation of acoustic energy fluxes generated in stellar convective zones (Goldreich 1987); in our case, however, it will emerge that this damping effect is negligible (see § VI).

In this paper, we derive the source function and energy flux for flux tube waves. We consider magnetic flux tubes embedded in a magnetic field-free, turbulent, and compressible medium and assume that the tubes are thin and oriented vertically; the latter assumption allows us to separate the generation of compressional tube waves from incompressional waves. In the present paper (Paper I), we concentrate on the generation of longitudinal tube waves, discuss the wave propagator and the relevant critical frequencies, and finally discuss the dependence of energy fluxes on the parameters which enter into the calculations. The generation of transverse tube waves and the application of the results to late-type stars will be treated in following papers.

The plan of our paper is as follows: The MHD equations and the basic formulation are presented in § II; the inhomogeneous wave equation and its solutions are given in § III; the energy fluxes for longitudinal tube waves are described in § IV; and the model parameter dependence of energy fluxes and their discussion are to be found in § V. A summary of our new results and our conclusions are given in § VI. Three appendices contain mathematical details, which amplify discussions in the main text.

II. MHD EQUATIONS AND THE BASIC FORMULATION

In this section, we discuss the basic equations of motion, and develop the formalism for calculating the generation rate for magnetic tube waves. In order to simplify the problem to the essentials, we shall assume that the fluid is locally isothermal, that the gas pressure is a scalar, and that displacement currents and electrostatic forces may be neglected; furthermore, it is straightforward to show that (as long as shock formation does not occur) dissipation by molecular viscosity and Ohmic diffusion is negligible for the problem at hand. In the following, we present the linearized magnethydrodynamic (MHD) wave equations, the basic assumptions for flux tubes, and finally the set of equations used to calculate the rate of tube wave generation.

a) The Linearized MHD Equations

Our assumptions lead to the ideal MHD equations, which may be written in the following linearized form:

$$\frac{\partial}{\partial t}\rho + \nabla \cdot (\rho_0 \boldsymbol{u}) = N_1 , \tag{2.1}$$

$$\frac{\partial}{\partial t} p + \boldsymbol{u} \cdot \nabla p_0 - V_s^2\left(\frac{\partial}{\partial t}\rho + \boldsymbol{u} \cdot \nabla \rho_0\right) = N_2 , \tag{2.2}$$

$$\frac{\partial}{\partial t}\boldsymbol{u} + \frac{1}{\rho_0}\nabla p - \frac{\rho}{\rho_0}\boldsymbol{g} - \frac{1}{4\pi\rho_0}[(\nabla \times \boldsymbol{B}) \times \boldsymbol{B}_0] = N_3 , \tag{2.3}$$

and

$$\frac{\partial}{\partial t}\boldsymbol{B} - \nabla \times (\boldsymbol{u} \times \boldsymbol{B}_0) = N_4 , \tag{2.4}$$

where ρ_0, p_0, B_0 refer to the unperturbed atmosphere, ρ, p, $\boldsymbol{u}$, $\boldsymbol{B}$ are the perturbations of density, pressure, velocity and magnetic field, respectively, $V_s(\equiv [\gamma RT/\mu]^{1/2})$ is the sound velocity and the N_i are nonlinear terms to be discussed below; the magnetic field is assumed to be potential, and the equation of motion is simplified by the assumption that the background atmosphere is in static equilibrium.

All terms linear in the perturbations in equations (2.1)–(2.4) are written on the left-handside; these terms define the wave propagation operators for MHD waves. However, terms quadratic in the perturbations are collected on the right-handside and are treated as known quantities described by a given flow (Lighthill 1952; Stein 1967). The latter terms determine the source function responsible for the MHD wave generation (Musielak and Rosner 1987) and are defined as follows:

$$N_1 = -\nabla \cdot (\rho \boldsymbol{u}) , \tag{2.5}$$

$$N_2 = -\boldsymbol{u} \cdot \nabla p + V_s^2 \boldsymbol{u} \cdot \nabla \rho , \tag{2.6}$$

$$N_3 = -\frac{\rho}{\rho_0}\frac{\partial}{\partial t}\boldsymbol{u} - \frac{\rho}{\rho_0}(\boldsymbol{u} \cdot \nabla)\boldsymbol{u} + \frac{1}{4\pi\rho_0}(\nabla \times \boldsymbol{B}) \times \boldsymbol{B} + \frac{1}{\rho_0}\boldsymbol{F}_g , \tag{2.7}$$

$$N_4 = \nabla \times (\boldsymbol{u} \times \boldsymbol{B}) , \tag{2.8}$$

where $\boldsymbol{F}_g$ is the fluctuating buoyancy force, which depends on the type of turbulence, and is responsible for the excitation of turbulent motions (Goldreich and Kumar 1988).

b) The Thin Flux Tube Approximation

In order to calculate the energy fluxes for flux tube waves, we assume that a vertically oriented magnetic flux tube is embedded in a stratified and nonmagnetized medium, and that all unperturbed and perturbed quantities depend on z and t alone; the z-axis is identified with the vertical direction (i.e., $\boldsymbol{g} = -g\hat{z}$) which, in our approach, is just the tube axis. We then obtain

$$\rho = \rho(z, t) + O(\epsilon) , \tag{2.9}$$

$$p = p(z, t) + O(\epsilon) , \tag{2.10}$$

$$\boldsymbol{u} = u_z(z, t)\hat{z} + O(\epsilon) , \tag{2.11}$$

$$\boldsymbol{B} = B_z(z, t)\hat{z} + O(\epsilon) , \tag{2.12}$$

where ϵ is defined as the ratio of the tube radius R_t to the tube length L_t; for $\epsilon \ll 1$, the tube can be treated in the thin flux tube approximation, an approximation which allows us to consider all unperturbed and perturbed quantities to zeroth order (see, for example, Roberts and Webb 1978). In this approach, the magnetic field within the flux tube is essentially axial, and is described by $\boldsymbol{B}_0 = B_0(z)\hat{z}$; note that the solenoidal condition does not restrict $B_0(z)$, but does, however, allow one to calculate the horizontal components of the magnetic field once the vertical component is known.

We further assume that the cross section of the tube is always circular, and that the thin flux tube is in temperature equilibrium with the surroundings, so that both density gradients inside $[d\rho_0(z)/dz]$ and outside $[d\rho_e(z)/dz]$ the tube are described by the same density scale height; hence, the vertical density variation is given by

$$\rho_{0,e}(z) = \rho_{00,ee} \exp\left(-\frac{z}{H_\rho}\right) , \tag{2.13}$$

where $H_\rho(\equiv V_s^2/\gamma g)$ is the density scale height, and is identical to the pressure scale height for an isothermal atmosphere.

In addition, if the total (gas plus magnetic) pressure is constant across the tube, then we must have (see Roberts and Webb 1978)

$$p_e(z) = p_0(z) + \frac{1}{8\pi} B_0^2(z) . \tag{2.14}$$

This condition allows us to obtain the vertical magnetic field variation, given by

$$B_0(z) = B_{00} \exp\left(-\frac{z}{H_b}\right) , \tag{2.15}$$

where the magnetic scale height H_b can be defined as

$$H_b = \frac{V_A^2}{g} \frac{\rho_0}{\rho_e - \rho_0} = 2H_\rho , \tag{2.16}$$

and where $V_A^2(\equiv B_0^2/4\pi\rho_0)$ is the square of the Alfvén velocity. In addition, equation (2.14) gives

$$\frac{B_0^2(z)}{B_0^2(0)} = \frac{p_0(z)}{p_0(0)} = \frac{p_e(z)}{p_e(0)} , \tag{2.17}$$

which allows us to easily calculate the structure of the tube when the boundary values for the pressure and magnetic field are given. Finally, we assume that the tube is untwisted, so that we ignore processes—such as the generation and propagation of torsional tube waves—as well as instabilities such as the kink instability. Such transverse wave motions will be considered in a subsequent paper.

c) *The Flux Tube Equations*

Combining the continuity and induction equations, and using equations (2.9)–(2.12) and (2.13)–(2.15), the set of MHD equations (2.1)–(2.4) can be rewritten as

$$\frac{\partial}{\partial t} \rho_1 - \frac{\partial}{\partial t} B_1 + \hat{W}_2 u_z = n_1 , \tag{2.18}$$

where n_1 contains the nonlinear terms and is given by $n_1 = -(1/\rho_0)N_1 - (1/B_0)N_{4z}$;

$$\frac{\partial}{\partial t} p_1 - V_s^2 \frac{\partial}{\partial t} \rho_1 + \hat{W}_g u_z = n_2 , \tag{2.19}$$

where $n_2 = (1/\rho_0)N_2$;

$$\frac{\partial}{\partial t} u_z + \hat{W}_1 p_1 + \rho_1 g = n_3 ; \tag{2.20}$$

where $n_3 = N_{3z}$; N_{3z} and N_{4z} are the z-components of N_3 and N_4 (eqs. [2.7] and [2.8]), respectively. In addition, $\rho_1 = \rho/\rho_0$, $p_1 = p/\rho_0$, and $B_1 = B_z/B_0$ are the new dimensionless perturbations; the operators $\hat{W}_1$, $\hat{W}_2$, and $\hat{W}_g$ are defined by

$$\hat{W}_1 \equiv \frac{\partial}{\partial z} - \frac{1}{H_\rho} , \qquad \hat{W}_2 \equiv \frac{\partial}{\partial z} - \frac{1}{2H_\rho} , \qquad \hat{W}_g \equiv (\gamma - 1)g . \tag{2.21}$$

The set of equations presented above (plus the horizontal pressure balance equation; see § II*d*) fully describes vertical motions inside the flux tube: The nonlinear terms n_1, n_2, and n_3 can be treated as the source of these motions and can in principle be determined once the driving turbulent flow in the tube interior is specified. As discussed below, we assume in our present (strong magnetic flux tube field) case that these internal sources are suppressed by the internal magnetic field, so that the wave motions inside the tube are driven from the outside alone. If there are no turbulent motions within the tube, the equations describe the propagation of longitudinal waves along the tube (see Roberts and Webb 1978 or Herbold *et al.* 1985 for comparison) and can be easily solved since all coefficients in these equations are constant. To close this set of equations, we need a relationship between the total pressure (gas and magnetic) inside the tube and the external pressure.

d) *The Horizontal Pressure Balance*

The relation between the variable fluid pressure applied to the undisturbed tube boundary by the external turbulence (or any other pressure variations) and the internal fluid pressure can be written as (Parker 1979, p. 218)

$$p_t(z, t) + \frac{1}{8\pi} B_t^2(z, t) = P_e(z, t) , \tag{2.22}$$

where p_t and B_t are the gas pressure and the magnetic field inside the tube, respectively. The pressure P_e outside the tube is defined by

$$P_e = p_e(z) + p_{\rm turb}(z, t) + p_{\rm te}(z, t) + p_w(z, t) , \tag{2.23}$$

where p_e is the gas pressure of the external fluid in the absence of any turbulence, and $p_{\rm turb}(z, t)$ is the pressure of the external turbulence. Note that there are two distinct pressure terms resulting from the backreaction of the tube on the external medium: $p_{\rm te}(z, t)$ denotes the external pressure perturbation due to the tube's back reaction because it is *displaced* from its mean position, while $p_w(z, t)$ is the external pressure perturbation due to the tube's back reaction because it is *deformed* from its mean shape. The former term disappears if the tube is at rest with respect to its surroundings; this term will be neglected in our calculations. The second term describes the excitation of acoustic waves in the outside medium due to cross-sectional variations of the tube and, as discussed by Spruit (1981), is unlikely to be important under solar conditions. Therefore, we shall neglect the interaction between longitudinal tube waves and external acoustic waves altogether. Thus, we may proceed to linearize equation (2.22); using equation (2.16), one then finds

$$p_1 + V_{\rm A}^2 B_1 = \frac{1}{\rho_0} p_{\rm turb} = n_4 , \tag{2.24}$$

where the variable fluid pressure $p_{\rm turb}$ caused by the external turbulence can be defined as

$$p_{\rm turb} \equiv \tfrac{1}{2}\rho_e(u_{tx}^2 + u_{ty}^2) , \tag{2.25}$$

and where u_{tx} and u_{ty} are the horizontal components of the turbulent velocity. We shall assume that these turbulent velocities can be determined from the specific turbulent flow outside the tube. Such an approach has been followed by Stein (1967) for the purely acoustic case, and by Musielak and Rosner (1987) for the case of homogeneous magnetic fields. Note that only motions directed toward to the tube wall contribute. As ρ_e/ρ_0 is of the order 10, $p_{\rm turb}/\rho_0$ can be considered to be a first-order term.

The set of tube equations (2.18)–(2.21), together with equation (2.24), fully describe the generation (due to the turbulent motions outside and possibly inside of the tube) and the propagation of longitudinal tube waves; this set of equations can be written as a single inhomogeneous wave equation and can be solved by Fourier-transforming this equation. This problem will be addressed in the next section.

III. THE INHOMOGENEOUS WAVE EQUATION

In this section, we derive and solve the inhomogeneous wave equation for longitudinal tube waves. We also calculate the source function and discuss the cutoff frequencies for longitudinal tube waves.

a) *The Wave Propagator*

We eliminate the density, magnetic field, and velocity perturbations from equations (2.18)–(2.21) and (2.24) and obtain the inhomogeneous wave equation for the pressure perturbations in the form

$$\left[\frac{\partial^2}{\partial t^2} - V_t^2 \frac{\partial^2}{\partial z^2} + 2V_t(\omega_{at} - \omega_{bt}) \frac{\partial}{\partial z} + \beta_e \omega_{ct}^2\right] p_1 = S_p(z, t) , \tag{3.1}$$

where the characteristic velocity for longitudinal tube waves (Defouw 1976) is given by

$$V_t^2 = \frac{V_{\rm A}^2 V_s^2}{V_{\rm A}^2 + V_s^2} , \tag{3.2}$$

and

$$\beta_e \equiv \frac{\rho_e}{\rho_0} = 1 + \frac{\gamma}{2} \frac{V_{\rm A}^2}{V_s^2} ; \tag{3.3}$$

we also define the three critical frequencies ω_{at}, ω_{bt}, and ω_{ct},

$$\omega_{at} \equiv \frac{V_t}{2H_\rho} , \qquad \omega_{bt} \equiv \frac{V_t}{2H_b} , \qquad \omega_{ct} \equiv \frac{V_t}{V_{\rm A}} \frac{g}{V_s} (\gamma - 1)^{1/2} = \frac{V_t}{V_{\rm A}} \omega_{\rm BV} , \tag{3.4}$$

where $\omega_{\rm BV}$ is the Brunt-Väisälä frequency. Note that for intense magnetic flux tubes ($V_s \ll V_{\rm A}$), ω_{ct} becomes the magnetic Brunt-Väisälä frequency; however, for $V_s \gg V_{\rm A}$, the characteristic tube velocity $V_t \approx V_{\rm A}$ and $\omega_{ct} \approx \omega_{\rm BV}$.

In equation (3.1), all the nonlinear terms are collected on the right-handside, where they become the source function for the longitudinal wave generation. We postpone the calculations of these source terms to the next subsection, and provide here only the definition of the source function

$$S_p(z, t) = \left(\frac{V_t}{V_s}\right)^2 \left\{\left(\frac{\partial^2}{\partial t^2} - g\hat{W}_2\right)\left[\left(\frac{\partial}{\partial t}\right)^{-1}(V_s^2 n_1 + n_2) + \left(\frac{V_s}{V_\mathrm{A}}\right)^2 n_4\right] - (V_s^2 \hat{W}_2 + \hat{W}_g)\left[n_3 - g\left(\frac{\partial}{\partial t}\right)^{-1} n_1 - \frac{g}{V_\mathrm{A}^2} n_4\right]\right\} . \tag{3.5}$$

To eliminate the first-order space derivative from the inhomogeneous wave equation (eq. [3.1]), we make the following transformation

$$p_1 = p_2\left(\frac{B_0}{\rho_0}\right)^{1/2} , \tag{3.6}$$

and obtain the inhomogeneous wave equation in the form

$$\left(\frac{\partial^2}{\partial t^2} - V_t^2 \frac{\partial^2}{\partial z^2} + \Omega_t^2\right)p_2 = S_t(z, t) , \tag{3.7}$$

where

$$S_t(z, t) = \left(\frac{\rho_0}{B_0}\right)^{1/2} S_p(z, t) \tag{3.8}$$

and

$$\Omega_t^2 = (\omega_{at} - \omega_{bt})^2 + \beta_e \omega_{ct}^2 ; \tag{3.9}$$

with help of equation (2.15), one obtains instead

$$\Omega_t^2 = \tfrac{1}{4}\omega_{at}^2 + \beta_e \omega_{ct}^2 . \tag{3.10}$$

Finally, after some algebraic manipulations, equation (3.10) can be written in the form given for the first time by Defouw (1976)

$$\Omega_t^2 = \frac{V_t^2}{H_\rho^2}\left(\frac{9}{16} - \frac{1}{2\gamma} + \frac{V_s^2}{V_\mathrm{A}^2}\frac{\gamma - 1}{\gamma^2}\right) . \tag{3.11}$$

The left-handside of equation (3.7) is the propagation operator for the tube waves; this relation allows for cross-sectional variations. In this case, the gas pressure is the principal restoring force. The waves which result are longitudinal tube waves, which may be viewed as acoustic waves propagating along the tube, but modified by the tube geometry. As a result of this modification, the characteristic phase velocity equation (3.2) and the critical frequency for the vertical propagation (eqs. [3.10] and [3.11]) are different from those for acoustic waves. By studying the dispersion relation (which is global in the approach considered here), it can be shown that Ω_t is the cutoff frequency for longitudinal tube waves, so that the waves cannot propagate if their wave frequency is either lower than or equal to this cutoff. As shown by equation (3.10), the cutoff frequency for longitudinal waves is not as simple as for acoustic waves and depends on both density and magnetic field scale heights, as well as on the Brunt-Väisälä frequency modified by the tube geometry; note, however, that the value of the critical frequency Ω_t is always comparable to the acoustic cutoff frequency (see eq. [3.11]). The tube cutoff frequency reaches a maximum when the tube structure is entirely dominated by the magnetic pressure (e.g., when the gas pressure inside the tube is negligible); in this limit, however, the longitudinal tube waves cannot propagate (this point will be further discussed in § V*b*).

b) The Source Function

To calculate the source function defined by equation (3.5), we make one additional assumption, namely that there are no turbulent motions inside the thin flux tube; this *Ansatz* simplifies our calculations substantially since $n_1 = n_2 = n_3 = 0$. Under this assumption, it is only the external turbulence which is responsible for the wave generation; this can be justified if we recall that strong magnetic fields (such as those characterizing the thin flux tube we are considering) will tend to suppress turbulent motions within the tube. Equation (3.8) can then be written in the following form

$$S_t(z, t) = \frac{\rho_e}{2(\rho_0 B_0)} \beta_t^2\left(\frac{\partial^2}{\partial t^2} + \omega_\mathrm{BV}^2\right)(M_{xx} + M_{yy}) , \tag{3.12}$$

where $M_{xx} \equiv u_{tx}^2$ and $M_{yy} \equiv u_{ty}^2$, and where $\beta_t \equiv V_t/V_\mathrm{A}$.

The source function does not depend either on the first or the second spatial derivative of M_{xx} and M_{yy} (see eq. [21] of Stein 1967); this does not mean, however, that we have pure monopole emission. As will be shown below, we actually have dipole emission as a result of our one-dimensional geometry. Finally, since the Brunt-Väisälä frequency is the buoyancy frequency (and thus the natural frequency for gravity waves), and as gravity waves are evanescent in the convectively unstable convection zone (where ω_BV^2 is negative), the term $\partial^2/(\partial t^2) + \omega_\mathrm{BV}^2$ constitutes a cutoff frequency for wave generation below ω_BV.

c) The Solution

Having obtained the source function, we Fourier-transform equations (3.7) and (3.12) in one dimension, using

$$[p_2(z, t), S_t(z, t)] = \iint dk'\, d\omega' [p_2(k', \omega'), \quad S_t(k', \omega')] \exp [i(\omega' t - k' z)] , \tag{3.13}$$

where

$$[p_2(k, \omega),\ S_t(k, \omega)] = \frac{1}{(2\pi)^2} \iint dz'\, dt' [p_2(z', t'),\ S_t(z', t')] \exp[-i(\omega t' - kz')] , \tag{3.14}$$

and obtain the solution for the pressure perturbations emitted by the turbulent motions,

$$p_2(k, \omega) = -\frac{S_t(k, \omega)}{\omega^2 - \Omega_t^2 - k^2 V_t^2} , \tag{3.15}$$

where $S_t(k, \omega)$ is the Fourier transform of the source function given by equation (3.12). To calculate the explicit form of the source function $S_t(k, \omega)$, we integrate equation (3.12) by parts, and obtain

$$S_t(k, \omega) = -\frac{1}{8\pi^2} \beta_e \left(\frac{\rho_0}{B_0}\right)^{1/2} \iint dz'\, dt' \beta_t^2 (\omega^2 - \omega_{\rm BV})(M'_{xx} + M'_{yy}) \exp[-i(\omega t' - kz')] . \tag{3.16}$$

The solution of the inhomogeneous wave equation given above will be used to calculate the energy fluxes emitted as a result of the interaction between the turbulent motions and the flux tube. Note that for $S_t(k, \omega) = 0$, the solution given by equation (3.15) leads to a dispersion relation which describes the propagation of longitudinal tube waves, and the critical frequency Ω_t becomes the cutoff frequency.

IV. ENERGY FLUXES FOR LONGITUDINAL TUBE WAVES

In this section, we derive the final expression for the energy fluxes emitted as longitudinal tube waves. We begin with the definition of energy flux for the flux tube geometry and then present and discuss the final results.

a) The Mean Energy Flux

The time-averaged longitudinal wave luminosity can be calculated by using the energy conservation principle, which gives

$$\langle L(z, t) \rangle = \langle \rho u_z A_t (\tfrac{1}{2} u_z^2 + W) \rangle , \tag{4.1}$$

where A_t is the cross-sectional area of the tube and W is the specific enthalpy; note that the wave luminosity defined above is the mean energy flux multiplied by the cross section of the tube and therefore has dimension (ergs s^{-1}). It should also be noted that within the thin flux tube approximation, the magnetic terms in equation (4.1) cancel one another, and that the energy flux does not depend on the magnetic energy; this reflects the fact that the gas pressure is the principal restoring force.

Expanding in an adiabatic perturbation series, and considering second-order quantities only, one obtains

$$\langle L(z, t) \rangle = \frac{5}{2} \frac{p_0}{\rho_0} A_0 \langle \rho u_z \rangle + \frac{5}{2} p_0 \langle A u_z \rangle + A_0 \langle p u_z \rangle , \tag{4.2}$$

where A is the perturbation of the tube cross section. The first term in equation (4.2) can be neglected as there is no net flow along the tube. One may estimate both remaining terms in equation (4.2) using the solutions given by Herbold *et al.* (1985),

$$\delta = \frac{5}{2} \frac{p_0 u_z A}{p u_z A_0} = \frac{5}{2} \frac{V_s^2}{\gamma V_{\rm A}^2} . \tag{4.3}$$

Finally, by combining equations (4.2) and (4.3), one obtains for the mean energy flux $F(z, t) = L(z, t)/A_0$

$$\langle F(z, t) \rangle = (1 + \delta) \langle p u_z \rangle . \tag{4.4}$$

Using the relation between the pressure perturbations p and p_2 (eq. [3.6]) and replacing u_z by u_z^*, the energy flux can be expressed in terms of p_2 and written in the form

$$\langle F(z, t) \rangle = (\rho_0 B_0)^{1/2} (1 + \delta) \langle p_2 u_z^* \rangle , \tag{4.5}$$

where the velocity perturbation u_z^* is calculated from equations (2.18)–(2.21) and 2.26), and is given by

$$u_z^* = -\left(\frac{B_0}{\rho_0}\right)^{1/2} \left[1 + \omega_{\rm BV}^2 \left(\frac{\partial}{\partial t}\right)^{-2}\right]^{-1} \left(\frac{\partial}{\partial t}\right)^{-1} \left(\frac{\partial}{\partial z} + \lambda\right) p_2^* , \tag{4.6}$$

where the asterisk denotes complex conjugate and

$$\lambda \equiv \frac{4 - 3\gamma}{4\gamma} \frac{1}{H_p} . \tag{4.7}$$

Now, we replace u_z^* in equation (4.5) by equation (4.6) and obtain

$$\langle F(z, t) \rangle = -B_0 (1 + \delta) \left\langle p_2 \left(\frac{\partial}{\partial t}\right)^{-1} \frac{(\partial/\partial z + \lambda)}{[1 + \omega_{\rm BV}^2 (\partial/\partial t)^{-2}]} p_2^* \right\rangle . \tag{4.8}$$

To evaluate the mean energy flux given by the above equation, we express $p_2(z, t)$ and $p_2^*(z, t)$ in terms of its Fourier transform (see eq. [3.13]), and obtain

$$\langle F(z, t)\rangle = B_0(1+\delta) \iiiint dk'\, dk''\, d\omega'\, d\omega''\, \omega'' \frac{k'' - i\lambda}{\omega''^2 - \omega_{\rm BV}^2} p_2(k', \omega') p_2^*(k'', \omega'') \exp\left[i(\omega' - \omega'')t - i(k' - k'')z\right] . \tag{4.9}$$

We take the time average over T_0 of the monochromatic wave energy by performing the integration over time t, and then obtain

$$\left\langle \frac{\partial}{\partial \omega} \langle F(z, t)\rangle \right\rangle_{T_0} = \frac{2\pi B_0}{T_0} (1+\delta) \frac{1}{V_t^4} \frac{\omega}{\omega^2 - \omega_{\rm BV}^2} I_{ss}(z, \omega) , \tag{4.10}$$

where

$$I_{ss}(z, \omega) = \iint dk'\, dk''(k'' - i\lambda) \frac{S_t(k', \omega)}{k'^2 - k_1^2} \frac{S_t^*(k'', \omega)}{k''^2 - k_1^2} \exp\left[-i(k' - k'')z\right] , \tag{4.11}$$

$$k_1^2 = \frac{\omega^2 - \Omega_t^2}{V_t^2} , \tag{4.12}$$

and where $\langle\ \rangle_{T_0}$ refers to a temporal average over a time scale T_0. Note that $\partial F(z, t)/\partial\omega$ does not have the standard energy flux dimension, but rather has dimensions (ergs cm^{-2} s^{-1} Hz^{-1}).

b) Monochromatic Wave Energy Flux

To obtain explicit forms for the energy fluxes emitted in the form of longitudinal waves, we must evaluate the asymptotic Fourier transform at large z (Appendix A) and calculate the product of the individual source functions and their conjugates. We transform the coordinates in equation (3.17) to average positions and times of the interacting turbulent eddies and then evaluate the turbulence velocity correlations by assuming that the fourth-order correlations can be reduced to second-order correlations (see Batchelor 1953, p. 23). We calculate the one-dimensional convolutions (Appendix B) over the turbulence spectrum by assuming the spectrum to be a separable product of a frequency-independent energy spectrum and a frequency factor (see § V*a* and Appendix B). Finally, after some algebra, we obtain the monochromatic wave energy flux (ergs cm^{-2} s^{-1} Hz^{-1}),

$$\frac{\partial F(z, t)}{\partial \omega} = \frac{8\pi^3 B_0}{T_0} (1+\delta) \frac{1}{V_t^4} \frac{\omega}{\omega^2 - \omega_{\rm BV}^2} \frac{k_1 - i\lambda}{k_1^2} |S_t(k_1, \omega)|^2 \sin^2(k_1 z - \phi) , \tag{4.13}$$

where ϕ is an arbitrary phase (henceforth assumed to be zero). The $\sin^2(k_1 z - \phi)$ term appears as a result of squeezing the tube at regular intervals of $z + n\pi/k_1$ along the tube; at these points, the efficiency of longitudinal wave generation is highest. In addition, $|S_t(k_1, \omega)|^2$ is the product of the individual source function and its conjugate and is given (from eq. [3.16]) by

$$|S_t(k_1, \omega)|^2 = \frac{T_0 Z_0}{2\pi^2} \frac{\rho_0}{B_0} \beta_e^4 \beta_t^4 (\omega^2 - \omega_{\rm BV}^2)^2 J_c(k_1, \omega) , \tag{4.14}$$

and where the convolution integral $J_c(k_1, \omega)$ is defined (Appendix B) by

$$J_c(k_1, \omega) = \frac{1}{16} \int_{-\infty}^{\infty} dk'\, E_2(k_1 - k') E_2(k') g(k_1, k', \omega) , \tag{4.15}$$

where k' is the wave number of a turbulent eddy. Following Hinze (1975, pps. 61, 202), we note that the one-dimensional energy spectrum $E_2(k')$ is connected to the three-dimensional energy spectrum $E(k)$ (see Appendix C) by the relation

$$E_2(k') = \frac{1}{2} \int_k^{\infty} dk \frac{E(k)}{k} \left(1 + \frac{k'^2}{k^2}\right) , \tag{4.16}$$

where k is the wave number of all turbulent motions which contribute to the one-dimensional energy spectrum. In addition, the function $g(k_1, k', \omega)$ is expressed by the frequency factor. Explicit forms for both the energy spectrum $E(k)$ and the frequency factor are given in § V*a* below.

The imaginary term in equation (4.13) can be neglected, as it vanishes upon integration over ω. In addition, we spatially average $\partial F(z, t)/\partial\omega$ over height $Z_0 \ll 1/k_1$ by performing a spatial integration of equation (4.13), and obtain the height-dependent contribution to the mean monochromatic wave energy flux (in units of ergs cm^{-3} s^{-1} Hz^{-1}) for a given flux tube,

$$\left\langle \frac{\partial^2 F(z, t)}{\partial \omega\, \partial z_0} \right\rangle = 2\pi(1+\delta)\rho_e \beta_e \beta_t^4 \frac{\omega(\omega^2 - \omega_{\rm BV}^2)}{V_t^3(\omega^2 - \Omega_t^2)^{1/2}} J_c(k_1, \omega) \sin^2(k_1 z) , \tag{4.17}$$

where a factor of $\frac{1}{2}$ was included to take into account the fact that only the outgoing flux is considered.

Separating the dimensional factors by using the turbulent velocity u_t and the turbulent length scale l_t, and performing the integration over z and ω, we obtain the total luminosity (ergs s^{-1}) due to longitudinal flux tube waves

$$L = N_t A_0 \int_0^H dz_0 \int_0^\infty d\omega \frac{\partial^2 F(z, t)}{\partial\omega \, \partial z_0}$$
$$= (2\pi)^4 N_t A_0 \int_0^H dz_0 \, \beta_e \beta_t^4 (1 + \delta) \frac{1}{l_t} \rho_e u_t^3 M_t^3 \int_0^\infty d\bar{\omega} \, \bar{\omega} \frac{(\bar{\omega}^2 - \bar{\omega}_{\rm BV}^2)}{(\bar{\omega}^2 - \bar{\Omega}_t^2)^{1/2}} \bar{J}_c(k_1, \bar{\omega}) \sin^2(k_1 z) , \quad (4.18)$$

where N_t is the number of flux tubes on the stellar surface, which can be related to the magnetic flux tube surface filling factor; $\rho_e u_t^3$ is (up to a constant of the order of unity) the convective flux (Ulmschneider and Stein 1982); and $M_t(\equiv u_t/V_t)$ is a coupling Mach number. The frequencies $\omega = \omega_t \bar{\omega}$ are written in terms of a characteristic turbulent frequency $\omega_t(\equiv 2\pi u_t/l_t)$ and a dimensionless frequency $\bar{\omega}$. Similarly, the convolution integral J_c is written as $u_t^3 l_t^2 \bar{J}_c$. Finally, H is the thickness of the turbulent region in the convection zone.

The total wave luminosity for longitudinal tube waves (eq. [4.18]) shows a dependence on the third power of the Mach number (dipole type of emission with respect to Mach number); this dipole source of emission can be explained by the process in which turbulent eddies moving in the opposite directions squeeze the flux tube and lead to the generation of the pressure perturbations (acoustic waves) that propagate along the magnetic field lines. The dipole nature of the longitudinal tube wave generation distinguishes our results from those obtained by Stein (1968), who considered the generation of acoustic waves in a nonmagnetic, stratified, and turbulent medium and found a dependence of the acoustic wave luminosity on the fifth power of the Mach number (quadrupole type of emission; see, however, Goldreich and Kumar 1988). The results presented here are also different from those given by Musielak and Rosner (1987), who obtained monopole type of emission for the generation of compressible MHD slow waves in a stratified medium with an embedded uniform magnetic field. Our results show, however, the same dependence on Mach number as that found by Goldreich and Kumar (1988) for forced turbulence in a nonmagnetized medium. This is no mere coincidence, but rather reflects the fact that the magnetic field acts very much like the external buoyancy force in constraining the motions of the fluid within the tube.

To calculate the wave luminosity given by equation (4.18), we have to specify the number of flux tubes on the stellar surface (e.g., the fraction of the stellar surface covered by flux tubes), as well as the values of the magnetic field strength, pressure, and density at the level in the atmosphere where the integrations begin. In addition, we must determine the wave frequency domain for longitudinal waves, and must describe the turbulence; in the latter case, we have to know the shape of the turbulence energy spectrum, the frequency factor, and the characteristic length scale of the turbulent motions. In the next section, we will show how these parameters are restricted by the observational data or by theoretical studies.

V. MODEL PARAMETER DEPENDENCE OF ENERGY FLUXES

In this section, we present preliminary energy fluxes and energy spectra for longitudinal waves, and discuss the dependence of our results on the free parameters. We begin with a description of the turbulence, discuss the dependence of the energy fluxes on the magnetic field and, finally, present the energy spectra for longitudinal tube waves and show how the results depend on the chosen magnetic field strength, the turbulent energy spectrum, and the frequency factor.

a) The Turbulence Energy Spectra and Turbulent Length Scale

Turbulent motions are characterized by the turbulent velocity and by the turbulent length scale; both these parameters are necessary to calculate the turbulent energy spectra, but have a simple form only for the very special case of isotropic, homogeneous, and incompressible turbulence. In this simple case, dimensional analysis shows that the frequency-independent energy spectrum $E(k)$ has the Kolmogorov form

$$E(k) = \frac{u_t^2}{k_t} \left(\frac{k}{k_t}\right)^{-5/3} , \qquad (k \geq k_t) \quad (5.1)$$

in the inertial range; alternatively, at smaller scales, where viscous effects begin to play a role, the spectrum takes on the exponential form

$$E(k) = \frac{64 u_t^2}{k_t} \left(\frac{k}{k_t}\right)^4 \exp\left(-\frac{4k}{k_t}\right) , \quad (5.2)$$

where the normalization condition for $E(k)$ is given by

$$\int_0^\infty E(k) dk = \frac{3}{2} u_t^2 , \quad (5.3)$$

and where the maxima of the Kolmogorov and exponential spectra occur near $k_t = 2\pi/l_t$.

Using the turbulence energy spectra given by equations (5.1) and (5.2), we can calculate the one-dimensional energy spectrum $E_2(k)$ defined by equation (4.16); and $E_2(k)$ can be evaluated analytically (see Appendix C). The results are presented in Figure 1, which shows a comparison between one-dimensional and three-dimensional turbulence energy spectra.

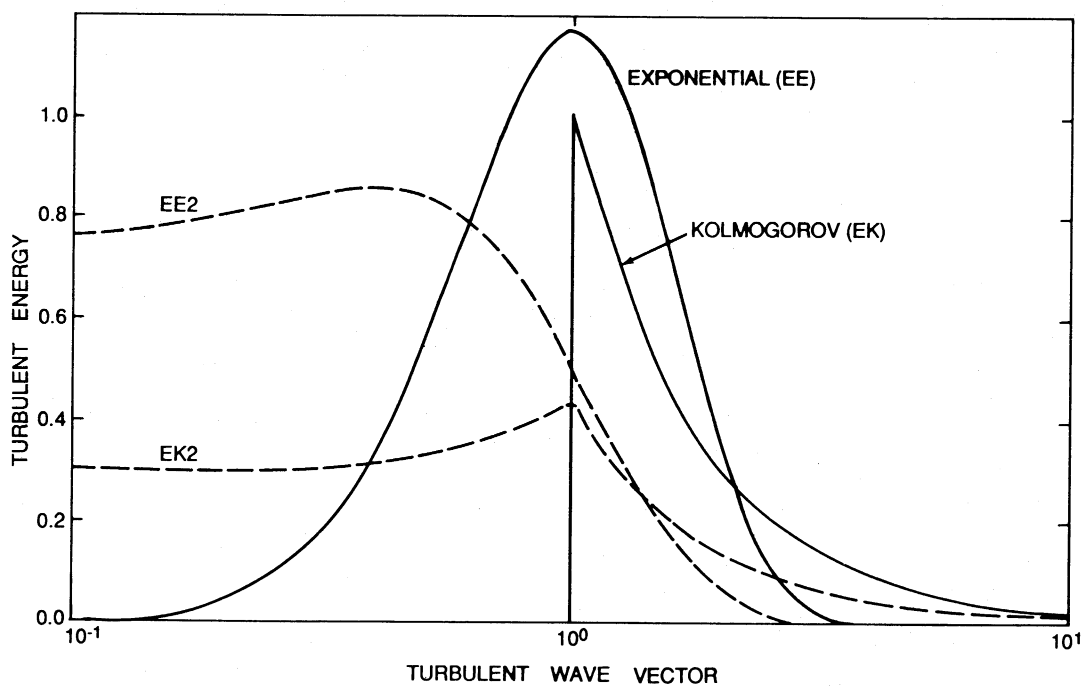

FIG. 1.—We display the one-dimensional (*dashed curves*) and three-dimensional (*solid curves*) dimensionless turbulent energy spectra (in units of $u_t^2 k_t^{-1}$) vs. the dimensionless turbulent wavevector (in units of k_t). We show $E_2(k)$ for an exponential turbulent energy spectrum (EE2) and for a Kolmogorov turbulent energy spectrum (EK2).

For the frequency factor, either the Gaussian form

$$\Delta(k, \omega) = \frac{2}{\pi^{1/2} k u^k} \exp\left[-\left(\frac{\omega}{k u_k}\right)^2\right], \tag{5.4}$$

or the exponential form

$$\Delta(k, \omega) = \frac{1}{k u_k} \exp\left[-\left(\frac{\omega}{k u_k}\right)\right], \tag{5.5}$$

can be assumed (see Stein 1967), with

$$u_k = \left[\int_k^{2k} E(k')dk'\right]^{1/2} . \tag{5.6}$$

In addition, $\Delta(k, \omega)$ is normalized so that

$$\int_0^{\infty} \Delta(k, \omega) d\omega = 1 . \tag{5.7}$$

The shape of the turbulence energy spectrum for stratified atmospheres is not known, and neither observational data nor theoretical considerations can help us to deduce its general properties. The energy spectra presented above certainly do not apply to the stratified turbulent atmosphere, especially to the largest turbulent eddies, whose size is comparable to the atmospheric scale height. Nevertheless, we shall adopt the above forms for the spectra since they span the range of likely behaviors of the actual spectra.

As an aside, we note that Stein (1967, 1968) and Bohn (1980) similarly adopted various ad hoc forms for the energy spectra, using the two functional forms just discussed and the so-called Spiegel turbulent spectrum, and three different functional forms for the frequency factors; in all these cases, stratified and magnetic field-free atmospheres were considered, and a turbulence correlation length scale equal to the pressure scale height was assumed. In addition, these authors assumed that the turbulent velocity of the largest eddies is the same as the velocity of convection motions given by the convection zone model. In our approach, the turbulent medium which surrounds a tube is stratified and magnetic field-free, so that in order to describe the turbulence, we have to make the same assumptions as those just mentioned. The difference is, however, that the largest turbulent eddies (with sizes comparable to the density or pressure scale height) do not contribute to the generation of longitudinal tube waves; in the interaction between the tube and turbulence, only eddies with sizes comparable to the tube diameter are important, and these eddies dominate longitudinal wave generation. We will include this effect in future energy flux calculations by computing the diameter of tube d_t for a given height (using the magnetic flux conservation law) and then, by setting $l_t = d_t$, estimate the turbulent velocity of the eddies with size d_t.

Unfortunately, neither the variations of the tube diameter with height in the deep photospheric layers, nor even the diameter of an individual "elementary" flux tube can be estimated with any confidence from the observational data because of current limitations on spatial resolution (see Solanki and Stenflo 1985).

b) *The Magnetic Field Strength*

In the approach presented in this paper, the magnetic field strength is a free parameter, but its maximum value is, in fact, fairly constrained by observational data, as well as by the horizontal pressure balance of the flux tube (see § II*d*). From the observational point of view, the magnetic field strength within flux tubes can be estimated for the Sun by analyzing the statistical properties of the Stokes *I* and *V* line profiles (Solanki and Stenflo 1985); this method gives the field strengths in typical network regions ranging from 1400 G to 1700 G. From a more theoretical perspective, horizontal pressure balance (based on the thin flux tube approximation) restricts the magnetic field pressure within the tube to be less than or equal to the gas pressure outside the tube (equality only obtains in the unlikely case that the flux tube is purely "magnetic," i.e., that there is no gas inside the tube). For typical published models of the solar photosphere (for example, Vernazza, Avrett, and Loeser 1983), the maximum field strength obtained from horizontal pressure balance also does not exceed 1700 G.

In our approach, the horizontal balance for pressure and magnetic field perturbations (eq. [2.26]) restrict p_1 and B_1 to be quantities of the same order as the turbulent pressure outside the tube; note that the turbulent pressure is a small fraction of the gas pressure outside the tube. If the gas pressure inside the tube becomes comparable to the turbulent pressure outside the tube, then perturbed and unperturbed quantities become comparable and the inhomogeneous tube wave equation (3.1) is no longer valid. In order to insure the validity of the perturbation scheme, we therefore consider only the cases when the unperturbed gas pressure inside the tube is at least twice as large as the turbulent pressure defined by equation (2.25); our method of calculation does not allow us to estimate the energy fluxes generated in the form of longitudinal waves for magnetic field strength values close to the maximum value.

c) *The Wave Energy Fluxes*

The wave energy fluxes presented in this paper are preliminary and are obtained for log $g = 4.5$, for one fixed tube diameter $l_t = d_t = 0.5H_\rho$, for one fixed Mach number $M \equiv u_t/V_s = 0.1$, for an external gas pressure varying from 11.0×10^5 to 5.0×10^5 dyn cm^{-2} and for magnetic field strengths varying from 1000 G to 1600 G; the latter variations of the magnetic field lead to variations of the tube Mach number from 0.15 to 0.08, respectively. These assumptions significantly simplify the problem and allow us to display the dependence of the wave luminosity spectra on the chosen magnetic field, the turbulent energy spectrum and the frequency factor. Note, however, that we do not need to specify the number of flux tubes on the stellar surface (or the filling factor) as we do not calculate the total wave luminosity.

The dependence of wave energy fluxes on the form of the chosen turbulent energy spectrum and the frequency factor are shown in Figures 2 and 3. As discussed above, we consider Kolmogorov and exponential energy spectra (eqs. [5.1] and [5.2]) and Gaussian and exponential frequency factors (eqs. [5.4] and [5.5]). In addition, these figures also show the dependence of the results on the magnetic field strength of $B_0 = 1500$ G and 1100 G; one sees a significant decrease in the wave energy flux when the magnetic field increases. This decrease of the wave energy flux with increasing magnetic field strength is mainly due to the $\beta_e \beta_t^4 \approx (V_A B_0)^{-2}$ term in equation (4.17), which describes the fact that as the tube background field strength increases, and the flux tube becomes increasingly rigid, it becomes more and more difficult for the external turbulence to excite gas pressure fluctuations within the tube.

The maxima of the wave energy fluxes occur at the wave frequency ($\omega \geq 2$–$4\Omega_t$) and, in general, for the Gaussian frequency factor are closer to Ω_t. The wave generation efficiency for the exponential turbulence spectrum is only slightly higher than for the Kolmogorov spectrum; the latter reflects small differences in the integrated one-dimensional energy spectrum $E_2(k)$ (eq. [4.16]) for both three-dimensional turbulent energy spectra considered here (see also Fig. 1).

Toward low frequencies, the wave energy flux rapidly decreases as the tube cutoff frequency Ω_t is approached. This effect is due to the $\omega^2 - \Omega_t^2$ term in the denominator of equation (4.17), and has already been discussed in § III*a* as a general property of our wave equation (3.7). At high frequencies, which after equation (4.12) are associated with large wave vectors k_1 the wave energy flux decays rapidly due to the strongly decreasing energy spectra and frequency factors. That is, if we compare the EE and KE spectra in Figure 3, we see that wave energy spectra derived with an exponential turbulent spectrum decay more steeply with frequency than those derived with a Kolmogorov turbulent spectrum; in addition, if we compare the EE and EG spectra in Figure 3, we see that wave energy spectra derived with a Gaussian frequency factor decay more steeply with frequency than those derived with an exponential frequency factor.

The dependence of the frequency-integrated energy fluxes on the magnetic field strength is given in Figure 4, which shows the results obtained for both turbulent energy spectra and both frequency factors. The results strongly depend on the magnetic field strength: the energy flux decreases when the magnetic field strength increases. This effect has already been discussed and simply reflects the increasing rigidity of the flux tube.

VI. DISCUSSION AND CONCLUSIONS

We have derived the source function and the energy fluxes for the generation of longitudinal tube waves in a thin, vertical flux tube embedded in an otherwise magnetic field-free stellar convection zone. We have shown that longitudinal tube waves in such flux tubes are generated by a dipole wave emission process; earlier qualitative results given by Stein (1981) and Ulmschneider and Stein (1982) suggested dominance of monopole emission (see, however, Goldreich and Kumar 1988). The similarity between the type of emission found here and that obtained by Goldreich and Kumar (1988) for forced turbulence in a nonmagnetized medium is apparently not accidental, as in both cases an effective "external" force exists which acts to constrain the motions of the radiating medium.

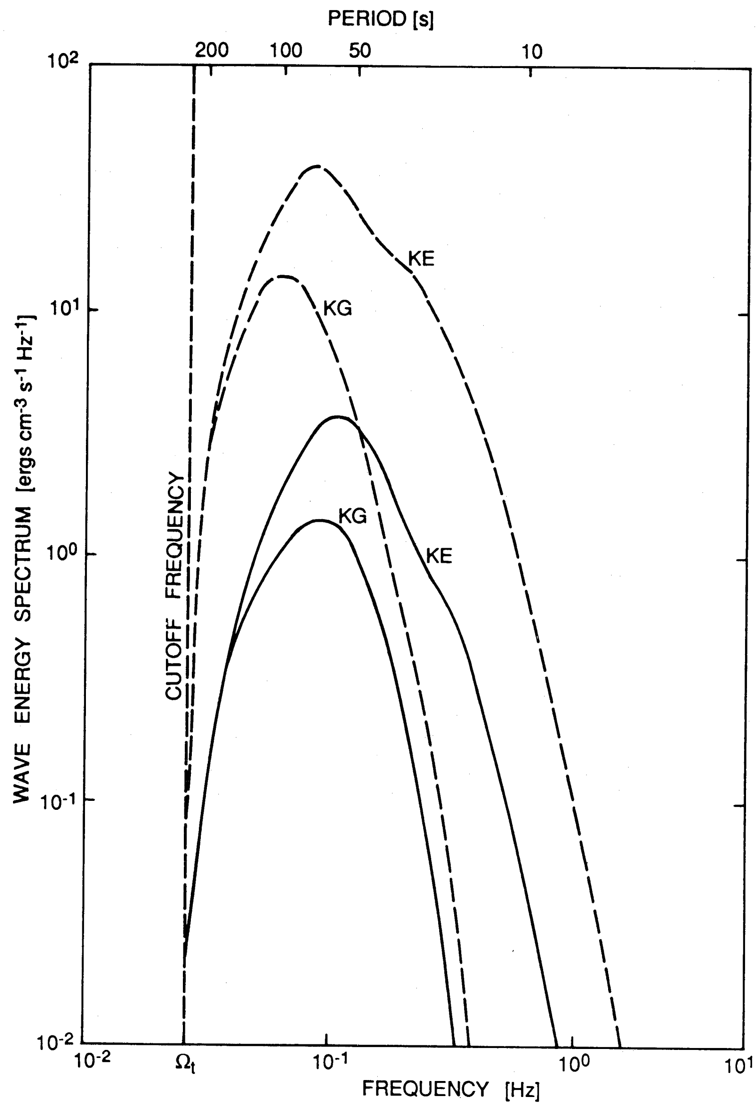

FIG. 2.—Comparison of the wave energy spectra vs. frequency for two different values of the magnetic field strength (B_0 = 1100 G, *dashed curves*; and = 1500 G, *solid curves*). The energy fluxes are obtained for a Kolmogorov turbulent energy spectrum and an exponential frequency factor (KE), and for an Kolmogorov energy spectrum and a Gaussian frequency factor (KG).

The efficiency of longitudinal wave generation decreases significantly when the magnetic field strength increases, mainly because the tube becomes increasingly rigid against the perturbations by the external turbulence. Note also that since the gas pressure within the tube decreases with increasing magnetic field strength, our approximations for the generation rate of longitudinal tube waves become unreliable as the gas pressure inside the tube approaches the turbulent pressure outside the tube. Furthermore, note that we have neglected the role of turbulence *within* the tube itself; to the extent that this assumption can be maintained, we were able to circumvent the difficulties encountered by previous workers who ignored the effects of the fluctuating buoyancy force.

The absorption of the generated waves also deserves some comment. The absorption results from the wave-turbulence interaction, and becomes important when the dimension of the region of wave generation is bigger than (or comparable to) the wavelength of the generated waves (Stein 1968). In our case, we find that both the vertical and horizontal dimensions of the emitting region (of the order of the thickness of the turbulent region, and of order of the tube diameter, respectively) are much smaller than vertical and horizontal wavelengths (viz., the latter is infinite for vertically propagating waves). As a result, damping due to a wave-turbulence interaction becomes negligible.

When acoustic waves are generated in a stratified and turbulent medium, monopole and dipole corrections to quadrupole emission also appear and show a narrow maximum in the energy flux for wave frequencies very close to the acoustic cutoff frequency (Stein 1967, Fig. 5); this energy flux cannot be carried away efficiently by acoustic waves from the generation region because these waves are almost evanescent. The situation is different for the forced turbulence case discussed by Goldreich and Kumar (1988), who showed that in this case dipole emissions dominate for a broad range of wave frequencies (see their eq. [17]). In the generation of longitudinal tube waves considered here, the maximum of the energy flux generated by dipole emission occurs at

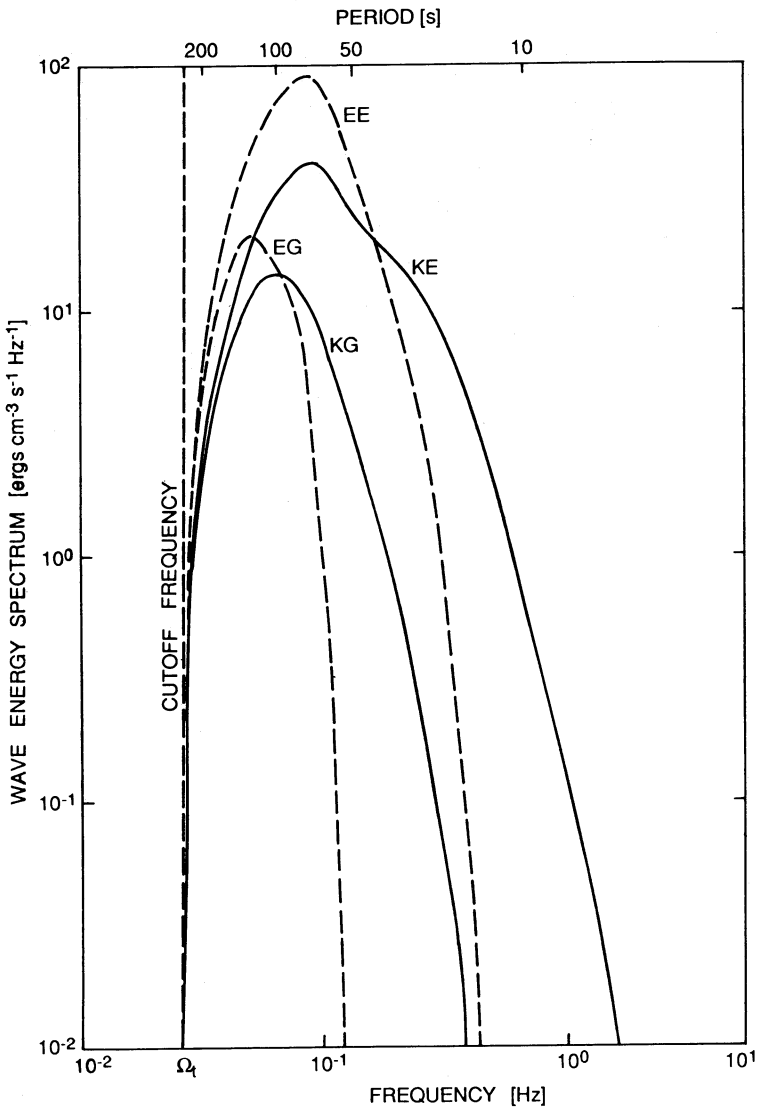

FIG. 3.—As in Fig. 2; we compare the effect of assuming a different spectral form for the turbulent energy spectra: the energy fluxes obtained a Kolmogorov turbulent energy spectrum are shown with solid lines (KG and KE), while those obtained with an exponential turbulent energy spectrum are shown with dashed lines (EG and EE). For the exponential turbulent energy spectrum, we denote the case of an exponential frequency factor by "EE," and for the case of a Gaussian frequency factor by "EG," In all cases, the magnetic field strength is $B_0 = 1100$ G.

$\omega = 2$–$4\Omega_t$ (making again our results more similar to those of Goldreich and Kumar) and then shows a steep decrease for higher frequencies.

The reason that the maximum of the dipole emission rate of longitudinal tube waves lies well above Ω_t is that the source function does not depend on the tube cutoff frequency, but instead depends on the Brunt-Väisälä frequency modified by the tube geometry. The latter frequency is lower than the tube cutoff frequency, which is comparable to the acoustic cutoff. Even if the acoustic cutoff frequency ω_{ac} (as well as the tube cutoff frequency Ω_t) increases in stellar photospheres (for example, $\omega_{ac} = 0.024$ at the top of the convection zone and $\omega_{ac} = 0.034$ at the temperature minimum; see Vernazza, Avrett, and Loesser 1976, model C), we may conclude that waves with frequencies near the maximum of the longitudinal wave energy flux reach the temperature minimum region as well as the chromosphere. Preliminary calculations of the energy fluxes for the flux tubes embedded in the solar convective zone (Musielak, Rosner, and Ulmschneider 1987) show, however, that these fluxes (being of order 10^{5-7} ergs cm^{-2} s^{-1}) are at least two orders of magnitude too low to play any role in the heating of the lower solar chromosphere. The fact that the wave energy fluxes are so low can be easily explained by two effects: first, the relatively slow (subsonic) turbulent motions in model solar and stellar convection zones; these motions (which in the present context totally determined the source function, and thereby the resulting wave energy fluxes) cannot produce substantial turbulent pressure perturbations, and therefore the interaction between the tube and the turbulence is of low efficiency. Second, recall that we have entirely suppressed turbulent motions *within* the flux tube; as a consequence, wave driving by buoyancy fluctuations cannot occur within the tube. Thus, judging by the results of Goldreich and Kumar (1988), it may well be that buoyancy fluctuations also dominate the wave generation problem for magnetic flux tubes; unfortunately, presently available theoretical tools are inadequate to study this problem in the strong field ($V_A \approx V_s$) limit.

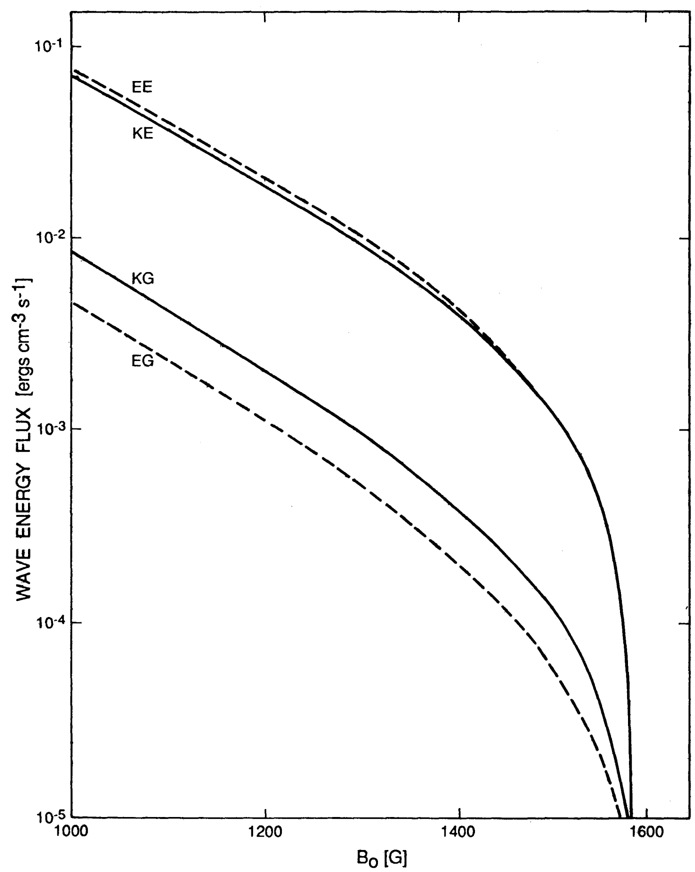

FIG. 4.—The frequency-integrated wave energy fluxes for longitudinal tube waves plotted as a function of the magnetic field strength for different turbulent energy spectra and different frequency factors; the annotations "KE," "KG," "EE," and "EG" are as in Figs. 3 and 3 and specify the functional form of the turbulent energy spectra (Kolmogorov [K] or exponential [E]) and the frequency factors (Gaussian [G] or exponential [E]).

Finally, we note that work corresponding to that discussed here remains to be carried out for transverse tube waves; this will be the aim of a succeeding paper.

We are very appreciative of P. Goldreich's extensive comments on the acoustic wave generation problem, and on our paper, and thank him for communicating recent unpublished results. We are indebted to G. Traving and T. Chang for discussions of turbulence-flux tube interaction problems, to S. Solanki for discussions of recent flux tube observational data, and to P. Gail for his help with the asymptotic Fourier transform and the turbulent energy spectrum. We also would like to thank D. Hathaway, R. Moore and S. Suess for reading and commenting on the manuscript. This work was supported in part by the Sonderforschungsbereich 132 while one of us (Z. E. M.) visited the University of Heidelberg during the summer of 1986, and by the NASA Solar-Terrestial Theory Program at the Harvard-Smithsonian Center for Astrophysics and at the University of Chicago (RR). The work was completed while Z. E. M. held a NRC-NASA/MSFC Research Associateship, with support from the NASA Solar and Heliospheric Physics Branch and Space Plasma Physics Branch.

APPENDIX A

THE ASYMPTOTIC FOURIER TRANSFORM

Let us consider a one-dimensional inhomogeneous partial differential equation with constant coefficients, given in the form

$$P\left(\frac{\partial^2}{\partial t^2}, \frac{\partial^2}{\partial z^2}\right)u(z, t) = f(z, t) , \tag{A1}$$

where P is a polynomial and $f(z, t)$ is a function which vanishes outside a restricted region. To solve this equation, we Fourier-transform $u(z, t)$ and $f(z, t)$, substitute these into equation (A1), and obtain

$$U(k, \omega) = \frac{F(k, \omega)}{P(-\omega^2, -k^2)} , \tag{A2}$$

where $U(k, \omega)$ and $F(k, \omega)$ are the Fourier transforms of $u(z, t)$ and $f(z, t)$, respectively. The solution for $u(z, t)$ can be written as

$$u(z, t) = \int_{-\infty}^{+\infty} d\omega \int_{-\infty}^{+\infty} dk \exp\left[-i(\omega t - kz)\right] \frac{F(k, \omega)}{P(-\omega^2, -k^2)} ; \tag{A3}$$

for the special case of a monochromatic source with frequency ω_0,

$$F(k, \omega) = F(k)\delta(\omega - \omega_0) , \tag{A4}$$

we have the result

$$u(z, t) = \exp(-\omega_0 t) \int_{-\infty}^{+\infty} dk \frac{\exp(ikz)F(k)}{P(-\omega_0^2, -k^2)} . \tag{A5}$$

Dropping the subscript zero, and since the integration over k can be performed with the help of the residue theorem, we find

$$u(z, t) = \exp(-\omega t) 2\pi i \sum_n \frac{\exp(ik_n z)F(k_n)}{[\partial G(k, \omega)/\partial k]\,|_{k_n}} , \tag{A6}$$

where $G(k, \omega) = P(-\omega^2, -k^2)$; see also Lighthill (1960).

Using equation (A6), we may evaluate the asymptotic Fourier transform of $I_{ss}(z, \omega)$ given by equation (4.11), and obtain

$$I_{ss}(z, \omega) = -\frac{\pi^2}{k^2}(k - i\lambda)[S_t^*(k, \omega)\exp(ikz) - S_t(k, \omega)\exp(-ikz)]^2 = 4\pi^2 \frac{k - i\lambda}{k^2} \,|S_t(k, \omega)|^2 \sin^2(kz - \phi) , \tag{A7}$$

where ϕ is an arbitrary phase.

APPENDIX B

THE CONVOLUTION INTEGRAL

In order to calculate the explicit form of the source function, it is necessary to evaluate a fourth-order velocity correlation. As this cannot as yet be done from first principles, we follow customary procedures (in the present context, see for example, Batchelor 1953 or Stein 1967), and replace the fourth-order correlation by a sum of products of second-order correlations:

$$\langle (u_x u_x + u_y u_y)(u'_x u'_x + u'_y u'_y) \rangle = 8 \langle u_x u_x \rangle \langle u'_x u'_x \rangle , \tag{B1}$$

where we have used the fact that there is no difference between the x- and y-directions. We thus have to evaluate the convolution integral

$$\begin{aligned} J_{xxxx}(k, \omega) &= \frac{1}{(2\pi)^2} \int_{-\infty}^{+\infty} dr \int_{-\infty}^{+\infty} d\tau \langle u_x(x, y, z, t) u_x(x, y, z + r, t + \tau) \rangle \langle u_x(x, y, z, t) u_x(x, y, z + r, t + \tau) \rangle \exp[i(\omega\tau - kr)] \\ &= \int_{-\infty}^{+\infty} dk' \int_{-\infty}^{+\infty} d\omega' \Phi_{xx}(k - k', \omega - \omega') \Phi_{xx}(k', \omega') , \end{aligned} \tag{B2}$$

where $\Phi_{xx}(k, \omega)$ is the Fourier transform of the velocity correlations.

We assume that $\Phi_{xx}(k, \omega)$ may be factored into the frequency-independent one-dimensional energy spectrum $E_2(k)$ and the frequency factor $\Delta(k, \omega)$

$$\Phi_{xx}(k, \omega) = \tfrac{1}{4} E_2(k) \Delta(k, \omega) , \tag{B3}$$

where k is the wave number of the kth eddy, and where the factor $\frac{1}{4}$ comes from the different normalization of $\Phi_{xx}(k, \omega)$ as compared to the energy spectrum $E_2(k)$ and the frequency factor Δ. Note that $E_2(k)$ is normalized so that

$$\int_0^{\infty} E_2(k) dk = u_t^2 \tag{B4}$$

and is connected to the three-dimensional energy spectrum (see Hinze 1975).

Using equation (B3), we may evaluate the convolution integral (B2) and obtain

$$J_{xxxx}(k, \omega) = \frac{1}{16} \int_{-\infty}^{\infty} dk' E_2(k - k') E_2(k') g(k, k', \omega) , \tag{B5}$$

where

$$g(k, k', \omega) = \int_{-\infty}^{+\infty} d\omega' \Delta(k - k', \omega - \omega') \Delta(k', \omega') . \tag{B6}$$

Because there is no difference between the x- and y-directions, we have

$$J_{xxxx}(k, \omega) = J_{xyxy}(k, \omega) = J_{yxyx}(k, \omega) = J_{yyyy}(k, \omega) = J_c(k, \omega) . \tag{B7}$$

APPENDIX C

THE ONE-DIMENSIONAL TURBULENT ENERGY SPECTRUM

The one-dimensional turbulent energy spectrum $E_2(k)$ defined by equations (4.16) can be evaluated analytically for both three-dimensional turbulent energy spectra considered here (eqs. [5.1] and [5.2]). We begin with the three-dimensional Kolmogorov energy spectrum, and calculate $E_2(k)$ by evaluating the integral in equation (4.16). It gives

$$E_2(k) = \frac{24}{55} \frac{u_t^2}{k_t^2} \left(\frac{k}{k_t}\right)^{-5/3} \qquad (k \geq k_t) , \tag{C1}$$

and

$$E_2(k) = \frac{3}{2} \frac{u_t^2}{k_t} \left(\frac{1}{5} + \frac{k^2}{11k_t^2}\right) \qquad (k < k_t) . \tag{C2}$$

For the exponential turbulence spectrum, the integral in equation (4.16) can be evaluated as

$$E_2(k) = \frac{u_t^2}{k_t} \left(\frac{16k^3}{k_t^3} + \frac{8k^2}{k_t^2} + \frac{3k}{k_t} + \frac{3}{4}\right) \exp\left(-\frac{4k}{k_t}\right) \tag{C3}$$

and is valid for all turbulent wavevectors k.

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