Wave pressure in stellar atmospheres due to shock wave trains

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Abstract. In the present work we derive analytic expressions for the wave pressure of propagating shock wave trains in stellar atmospheres or winds. Applications to weak shocks and stronger shocks with sawtooth profiles are discussed in detail. The shocks are treated as discontinuities. Our results provide insight in the momentum balance of time-dependent stellar winds flows. The analytic expressions can be used as an independent test of hydrodynamic codes.

Key words: Stellar atmospheres – chromospheres – shocks – late-type stars

1. Introduction

Wave pressure is self-consistently included in time-dependent wave computations when the hydrodynamic equations are solved and the shocks are treated explicitly. The need to consider wave pressure separately arises only, when the momentum equation is solved in time-independent computations of stellar atmosphere or wind models. Recent time-dependent calculations of acoustically heated outer atmosphere models of late-type stars (see e.g. Ulmschneider, 1989) have again pointed out the well known limiting shock strength behaviour of monochromatic waves in gravitational atmospheres. This behaviour of shocks to reach a limiting strength arises from the balance of two opposing tendencies: Due to wave energy conservation, the rapid density decay in the stellar atmosphere leads to an amplitude growth, while on the other hand, shock dissipation results in a decrease of the wave amplitude. The validity of the limiting shock strength behaviour in wave computations where hydrogen ionization, radiation damping, shock dissipation and variable gravity are taken into account has been discussed by Cuntz and Ulmschneider (1988).

In monochromatic and stochastic short period wave models for the outer atmosphere of Aldebaran, Cuntz (1990) has recently investigated the influence of wave pressure on time-dependent stellar atmospheric structures. By time-averaging he evaluated the local actual pressure scale height and the local thermal pressure scale height in the models. The actual pressure scale height includes the full influence of the waves on the atmospheric structure while the thermal pressure scale height incorporates just the change of temperature and mean molecular weight. Cuntz found that the two scale heights are quite similar for monochromatic waves with small amplitudes. For stochastic waves, however, the actual pressure scale height becomes significantly larger than the thermal pressure scale height demonstrating the importance of the wave pressure by strong shocks.

The limiting strength property of the waves produces wave trains with rather typical and universal shape (see e.g. Ulmschneider, 1989) which is independent of the source of the acoustic energy generation. This opens the possibility to construct time-independent shock heated atmospheric models. The waves change the atmospheric model structure through their contribution in the momentum and energy equations. The momentum contribution is the wave pressure and the energy contribution is shock dissipation and radiation damping. For the energy contribution, heating formulae are already available in the literature (see e.g. Ulmschneider, 1970; Bray and Loughhead, 1974). Time-independent wave models can be calculated if both contributions are considered consistently. Such models give general insight in the behaviour of the waves and can easily be used as starting models for time-dependent computations.

There are older time-independent atmospheric model computations (e.g. McWhirter et al., 1975) which have used a wave pressure term, but due to an inadequate expression (propagation speed taken equal to the sound speed) for this term difficulties were encountered. Hartmann and McGregor (1980) have considered wave pressure in their Alfven wave driven wind models computing the momentum transfer of the waves from their energy deposition. Further attempts to include wave pressure in time-independent wave models were discussed by Holzer and McGregor (1985) for various heating mechanisms. In the present work we want to obtain useful expressions for the wave pressure from shock wave trains. For this purpose, we derive, in Sect. 2, a general expression for the average momentum deposition by the shocks. In Sect. 3 we quantify this expression for weak shocks and larger amplitude shocks with sawtooth profiles. Section 4 gives our conclusions.

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2. Momentum transfer of shock waves

We consider acoustic shock waves which travel through a stellar atmosphere. If there is only one shock (or at most a few shocks) present at any one time, it is most natural to consider the shock propagation through the atmosphere as a time-dependent problem. If, on the other hand, at any one instant, there exists a large number of shocks within the atmosphere, it will often be sufficient to consider only the average response of the atmosphere on the presence of an ensemble of shocks. In the following, we derive an equation for the average structure of such an atmosphere by taking the time-average of the momentum balance.

2.1. Time average of the total momentum

Let us consider some arbitrary but fixed volume element \( V \) within the stellar atmosphere. The total momentum of the gas contained in \( V \) is given by

\[
P_\mu = \int_V dV \rho v_\mu.
\]

Here the index \( \mu \) indicates the different vector components. If we observe this volume element for a sufficiently long time interval \( T \), many shocks traverse \( V \). The momentum \( P_\mu \) of the gas contained at each instant in \( V \) varies strongly with time and so does the time derivative \( \partial P_\mu / \partial t \) which equals the instantaneous net force exerted on the gas. The average net force exerted during the period \( T \) on the gas then is given by

\[
\langle \dot{P}_\mu \rangle = \frac{1}{T} \int_0^T dt \frac{\partial P_\mu}{\partial t}.
\]

![Fig. 1. Definition of the instants \( t_i \) and \( t_{i+1} \) when a shock enters or leaves \( V \), respectively.](image)

For calculating this average, we assume \( V \) to be small enough that there is at most one shock in \( V \). We denote by \( t_i \) the instants, at which a shock enters into \( V \) and by \( t_i \) the instants, at which a shock leaves \( V \) (cf. Fig. 1). The index \( i \) enumerates the shocks traversing \( V \) during the period \( T \). The total number of shocks crossing \( V \) during the time interval \([0, T] \) is denoted by \( I \). We choose \([0, T] \) such that we have \( t_1 = 0 \) and \( t_{I+1} = T \), i.e., we start the averaging process at the instant where a shock enters into \( V \) and finish it at the instant where the \((I + 1)\)-th shock enters \( V \).

We have to consider two different cases:

1.) There is no shock in \( V \). This holds during the \( I \) intervals of time \([t_i, t_{i+1}] \).

2.) There is one shock in \( V \). This holds during the \( I \) intervals of time \([t_i, t_{i+1}] \).

These cases have to be treated differently in calculating the average momentum transfer.

In the first case during the time intervals between shocks we have

\[
\frac{1}{T} \int_{t_i}^{t_{i+1}} dt \dot{P}_\mu = \frac{1}{T} \int_{t_i}^{t_{i+1}} dt \int_V d\xi \frac{\partial \rho v_\mu}{\partial t}.
\]

Since no shocks are present, all hydrodynamic quantities are continuously differentiable and there holds the usual equation of momentum conservation

\[
\frac{\partial \rho v_\mu}{\partial t} + \frac{\partial}{\partial \xi} \left( \rho v_\nu v_\mu + p \delta_{\mu \nu} \right) = -g_\mu \rho
\]

where \( g_\mu \) is the inwardly directed gravitational acceleration. Note that as customary we assume summation if the same component index occurs twice in an expression. Inserting this into Eq. (3), observing that \( V \) is a fixed volume and interchanging integrations and differentiation (permitted, since all quantities are continuously differentiable) yields

\[
\frac{1}{T} \int_{t_i}^{t_{i+1}} dt \frac{\partial P_\mu}{\partial t} =
\]

\[
- \frac{1}{T} \int_V dV \left\{ \frac{\partial}{\partial \xi} \int_{t_i}^{t_{i+1}} dt \left[ \rho v_\nu v_\mu + p \delta_{\mu \nu} \right] + g_\mu \int_{t_i}^{t_{i+1}} dt \rho \right\}.
\]

On the other hand, we have

\[
\frac{1}{T} \int_{t_i}^{t_{i+1}} dt \dot{P}_\mu = \frac{1}{T} \left[ P_\mu(t_{i+1}) - P_\mu(t_i) \right].
\]

Hence it follows for the intervals \([t_i, t_{i+1}] \) between shocks

\[
\frac{1}{T} \left[ P_\mu(t_{i+1}) - P_\mu(t_i) \right] =
\]

\[
- \frac{1}{T} \int_V dV \left\{ \frac{\partial}{\partial \xi} \int_{t_i}^{t_{i+1}} dt \left[ \rho v_\nu v_\mu + p \delta_{\mu \nu} \right] + g_\mu \int_{t_i}^{t_{i+1}} dt \rho \right\}.
\]

In the second case, for all \( t \in [t_i, t_{i+1}] \) there is one shock in \( V \). We denote all quantities ahead of the shock by an index 1 and all quantities behind the shock by an index 2 (cf. Fig. 2). The shock can be considered as an infinitely thin layer in \( V \), dividing the volume \( V \) in a pre-shock volume \( V_1 \) and a post-shock volume \( V_2 \). Since the jumps of the hydrodynamic quantities across the shock are bounded, any contribution of a spatial integration over the infinitely thin shock layer to the volume integration over \( V \) can be neglected. Then

\[
P_\mu = \int dV (\rho v_\mu)_2 - \int dV (\rho v_\mu)_1
\]

with

\[
V_1(t) + V_2(t) = V.
\]
The volumes \( V_1(t) \) and \( V_2(t) \) depend on the time \( t \) since the shock moves across \( V \). According to our definitions we have

\[
\begin{align*}
V_1(t_i) &= V, & V_2(t_i) &= 0 \\
V_1(t) &= 0, & V_2(t) &= V
\end{align*}
\tag{10}
\]

and

\[
\begin{align*}
P_\mu(t_i) &= \int_V dV \left( \rho v_\mu \right)_1 \\
P_\mu(t) &= \int_V dV \left( \rho v_\mu \right)_2.
\end{align*}
\tag{11}
\]

\( P_\mu(t) \) is a continuously differentiable function of the time \( t \in [t_i, t_{i+1}] \), since this holds for \( V_1(t) \) and \( V_2(t) \) as well as for the integrands within \( V_1 \) and \( V_2 \). Thus, it follows

\[
\frac{1}{T} \int_{t_i}^{t_{i+1}} dt \, \dot{P}_\mu = \frac{1}{T} \left[ P_\mu(t_{i+1}) - P_\mu(t_i) \right]
\tag{12}
\]

and from Eqs. (11) we obtain for the intervals \([t_i, t_{i+1}]\) where shocks are present in \( V \):

\[
\frac{1}{T} \int_{t_i}^{t_{i+1}} dt \, \dot{P}_\mu = \frac{1}{T} \int_V dV \left[ \left( \rho v_\mu(t_{i+1}) \right)_2 - \left( \rho v_\mu(t_i) \right)_1 \right].
\tag{13}
\]

Note that the spatial integration is over the post–shock momentum at instant \( t_i \) and over the pre-shock momentum at instant \( t_i \), respectively.

2.2. Average momentum change

By combining Eqs. (6) and (12) for all \( I \) intervals with and without shocks, we obtain

\[
\langle \dot{P}_\mu \rangle = \frac{1}{T} \int_0^T dt \, \dot{P}_\mu = \frac{1}{T} \left[ P_\mu(T) - P_\mu(0) \right]
\tag{14}
\]

since the contributions of all division points \( t_i \) and \( t_{i+1} \) of the \( t \)-axis between \( t = 0 \) and \( t = T \) cancel pairwise. On the other hand, by combining Eqs. (7) and (13) for all our intervals, we obtain by means of (14)

\[
\frac{1}{T} \left[ P_\mu(T) - P_\mu(0) \right] =
\]

\[
-\frac{1}{V} \int dV \left\{ \frac{\partial}{\partial X_\nu} \sum_{i=1}^{I_{i+1}} \int_{t_i}^{t_{i+1}} dt \left[ \rho v_\nu v_\mu + p \delta_\mu_\nu \right] + g_\mu \sum_{i=1}^{I_{i+1}} \int_{t_i}^{t_{i+1}} dt \rho \right\}
\]

\[
+ \frac{1}{V} \int dV \sum_{i=1}^{I} \left[ \left( \rho v_\mu(t_i) \right)_2 - \left( \rho v_\mu(t_i) \right)_1 \right].
\tag{15}
\]

We divide this equation by the volume \( V \) and take the limit of an infinitely small volume element \( V \). In this limit the volume integrals divided by \( V \) tend to the value of the integrands at the place of the infinitely small volume element and simultaneously the instants \( t_i \) become identical with the instants \( t_i \). The term in brackets in the second term in Eq. (15) becomes equal to the jump \( \left[ \left( \rho v_\mu \right)_2(t_i) - \left( \rho v_\mu \right)_1(t_i) \right] \) of \( \rho v_\mu \) across the shock front. It follows

\[
\frac{1}{T} \left[ \left. \rho v_\mu \right|_{t=T} - \left. \rho v_\mu \right|_{t=0} \right] = -\frac{\partial}{\partial X_\nu} \left[ \langle \rho v_\nu \rangle_v + \langle p \rangle \delta_\mu_\nu \right] - g_\mu \langle \rho \rangle
\]

\[
+ \frac{1}{T} \sum_{i=1}^{I} \left[ \left( \rho v_\mu \right)_2(t_i) - \left( \rho v_\mu \right)_1(t_i) \right]
\tag{16}
\]

where

\[
\langle A \rangle = \frac{1}{T} \int_{t_i}^{t_{i+1}} \int dt A(t)
\tag{17}
\]

is the time average of some quantity \( A \) between shocks.

The last term on the r.h.s. of Eq. (16) is the average of the momentum jump \( \rho v_\mu \) across the shock front. This momentum is transferred to the gas. Let \( \vec{k}_i \) be the normal of the \( i \)-th shock front. Then

\[
(\rho v_\mu)_2(t_i) - (\rho v_\mu)_1(t_i) = \left[ \langle \rho v_\mu \rangle_2 - \langle \rho v_\mu \rangle_1 \right] \hat{k}_i
\tag{18}
\]

where \( v \) is the velocity component of the gas in the direction of \( \vec{k}_i \). We denote the velocity of the \( i \)-th shock front in the stellar frame of rest by \( U_i \) and the pre-shock and post-shock velocity components of the gas normal to the shock front in the frame where the shock front is at rest by \( u_{i,1} \) and \( u_{i,2} \), respectively. Then

\[
v_{i,1} = U_{i,1} + U_i, \quad v_{i,2} = U_{i,2} + U_i
\tag{19}
\]

and from the Hugoniot condition,

\[
\langle p\mu \rangle_2 = \langle p\mu \rangle_1,
\tag{20}
\]

of mass conservation across the shock we obtain

\[
(\rho v_\mu)_2(t_i) - (\rho v_\mu)_1(t_i) = \left[ \rho_{i,2} - \rho_{i,1} \right] U_i \hat{k}_i
\tag{21}
\]

We define the mean time between shocks by

\[
\frac{\langle \rho \rangle}{T_I} = \frac{T}{I}
\tag{22}
\]

and the average momentum transfer of a shock to the gas by

\[
\frac{1}{I} \sum_{i=1}^{I} (\rho_{i,2} - \rho_{i,1}) U_i \hat{k}_i = \langle \rho_2 - \rho_1 \rangle U_i \hat{k}_i.
\tag{23}
\]
Further, we note that for a sufficiently long time $T$ the left-hand side of Eq. (16) becomes negligibly small if the pre-shock momenta $p_{\mu}$ at $t = 0$ and $t = T$ on the average vary slower than linear with $T$, which usually will be the case in a stationary atmosphere. Then it follows that

$$\frac{\partial}{\partial x_{\nu}} \left[ \langle \rho \rangle \delta_{\mu\nu} + \langle p_{\mu} v_{\nu} \rangle \right] - \frac{1}{\mathcal{P}} \langle p_{2} - p_{1} \rangle U \hat{k}_{\nu} = -g_{\mu}(\rho). \quad (24)$$

This is our final result for the average momentum balance in an atmosphere with many shocks. The second and third terms on the left-hand side of Eq. (24) represent the wave pressure. The second term, which is always different from zero in an atmosphere with non-vanishing fluid motions, is present even in the absence of shocks. This represents the usual wave pressure of an ensemble of sound waves. The third term on the left-hand side represents the average momentum transfer of shocks to the gas which enters into the equation like an external force. Its form can easily be understood by the following consideration assuming periodic shock waves. Let $u_k$ and $u_d$ be the velocity in front of and behind a shock of speed $U$. Then $p_2 u_2 - p_1 u_1 = (p_2 - p_1) U$ is the change in momentum density at a fixed position following shock passage. If the interval in time between successive shocks is $\mathcal{P}$, then with the passage of each shock, a parcel of gas experiences an impulsive force $(p_2 - p_1) U / \mathcal{P}$.

$$\langle p_{\mu} v_{\nu} \rangle = a(z) \delta_{\mu\nu} + b(z) \hat{n}_{\mu} \hat{n}_{\nu}. \quad (25)$$

$a$ and $b$ may depend on the coordinate $z$ parallel to $g_{\mu}$. By contracting Eq. (25) and by multiplying Eq. (25) with $\hat{n}_{\mu} \hat{n}_{\nu}$ one easily shows

$$\langle p_{\mu} v_{\nu} \rangle = \frac{1}{2} \langle p v_{\nu} \rangle \delta_{\mu\nu} + \left\{ \langle p v_{\nu} \rangle - \frac{1}{2} \langle p v_{\nu} \rangle \right\} \hat{n}_{\mu} \hat{n}_{\nu} \quad (26)$$

where $v_1$ and $v_d$ are the velocity components perpendicular and parallel to $\hat{n}$, respectively. Since the averages $\langle p v_{\nu} \rangle$ and $\langle p v_{\nu} \rangle$ depend only on the $z$-coordinate,

$$\frac{\partial}{\partial x_{\nu}} \langle p_{\mu} v_{\nu} \rangle = \hat{n}_{\mu} \frac{\partial}{\partial z} \langle p v_{\nu} \rangle. \quad (27)$$

For the same symmetry reasons, the average momentum transfer of shocks is a vector parallel to $g_{\mu}$, thus

$$\langle (p_2 - p_1) U \hat{k} \rangle = \langle (p_2 - p_1) U \hat{k} \cdot \hat{n} \rangle \hat{n}. \quad (28)$$

Combining Eqs. (24) to (28) yields

$$\frac{\partial}{\partial z} \langle p \rangle + \frac{\partial}{\partial z} \langle p v_{\nu} \rangle - \frac{1}{\mathcal{P}} \langle (p_2 - p_1) U \hat{k} \cdot \hat{n} \rangle = -g(\rho), \quad (29)$$

which is valid for the average momentum balance in a plane parallel stellar atmosphere.

3. Wave pressure for different shock trains

In this section, we derive expressions for the wave pressure for the case of vertically propagating shock wave trains. A typical example of a shocked atmosphere is shown in Fig. 3 (Cuntz, 1987). This result of a numerical solution of the full system of hydrodynamic equations demonstrates clearly that the variation of the hydrodynamic quantities between shocks can rather accurately be approximated by linear segments (except, perhaps, for the very strongest shocks). For this reason, in the following, we restrict our considerations to sawtooth shock waves. We first consider weak shocks and then somewhat larger amplitude shocks, where the velocity amplitude is still considerably smaller than the sound speed. The time-averaged atmosphere is assumed to be at rest.

3.1. Weak shocks

We consider an ensemble of weak shocks propagating outwards parallel to $g_{\mu}$ in a plane-parallel stellar atmosphere. For small amplitude sawtooth waves one has the following velocity, density and pressure variations at a fixed point within the atmosphere

$$v(t) = v_{m,i} - 2v_{m,i} t \frac{\partial}{\partial t}, \quad (30)$$
\[\rho(t) = \rho_0 + \rho_{m,i} - 2 \rho_{m,i} \frac{t}{\mathcal{P}_i},\]  \hspace{1cm} (31)

\[p(t) = p_0 + p_{m,i} - 2 p_{m,i} \frac{t}{\mathcal{P}_i},\]  \hspace{1cm} (32)

where \(\mathcal{P}_i = t_i - t_{i-1}\) is the time between passage of the \((i-1)\)th and \(i\)th shocks, subscript \(m\) indicates maximum amplitudes, and \(\rho_0\) and \(p_0\) are the unperturbed density and pressure, respectively. We assume that \(v_0 = 0\), i.e., that no average flow occurs in the atmosphere. Then one has

\[\langle p \rangle = \frac{1}{T} \sum_{i=1}^{l} \int_0^{\mathcal{P}_i} dt \, p(t) = \frac{1}{T} \sum_{i=1}^{l} \rho_0 \mathcal{P}_i = \rho_0\]  \hspace{1cm} (33)

since \(\sum_i \mathcal{P}_i = T\) and correspondingly

\[\langle \rho \rangle = \rho_0.\]  \hspace{1cm} (34)

For the second term at the l.h.s. of Eq. (29) we obtain

\[\langle p v_i^2 \rangle = \frac{1}{T} \sum_{i=1}^{l} \int_0^{\mathcal{P}_i} dt \, \rho(t) v_i^2(t) = \frac{1}{T} \rho_0 \sum_{i=1}^{l} v_{m,i}^2 \mathcal{P}_i.\]  \hspace{1cm} (35)

For weak shocks one has for the velocity jump across the shock front

\[v_{m,i} = \frac{1}{2} c_S \eta_i,\]  \hspace{1cm} (36)

where

\[\eta_i = \frac{\rho_2 - \rho_1}{\rho_1} \ll 1\]  \hspace{1cm} (37)

is the shock strength and \(c_S\) the sound speed (Ulmschneider, 1970). \(\rho_1, \rho_2\) are the densities in front and behind the shock. From Eqs. (35) to (37) it follows that

\[\langle p v_i^2 \rangle = \frac{1}{12} \rho_0 c_S^2 \eta_i^2,\]  \hspace{1cm} (38)

where

\[\langle \eta_i^2 \rangle = \sum_{i=1}^{l} \eta_i^2 \frac{\mathcal{P}_i}{T}.\]  \hspace{1cm} (39)

is the square average of the shock strength. The second term at the l.h.s. of Eq. (29) can then be written as a wave pressure force

\[f_w = \frac{d}{dz} \frac{1}{12} \rho_0 c_S^2 \langle \eta_i^2 \rangle\]  \hspace{1cm} (40)

or, alternatively, as

\[f_w = d \frac{F_{\text{mech}}}{dz} c_S,\]  \hspace{1cm} (41)

since \(\gamma \rho_0 = c_S^2 / \gamma g\) and

\[F_{\text{mech}} = \frac{1}{12} \rho_0 c_S^2 \langle \eta_i^2 \rangle.\]  \hspace{1cm} (42)

Eq. (42) represents the mechanical energy flux of sawtooth shock waves (Ulmschneider, 1970, note difference of notation of \(F_{\text{mech}}\)). For weak shocks, the dissipation of \(F_{\text{mech}}\) with height is given by

\[\frac{d F_{\text{mech}}}{dz} = -\frac{1}{12} \gamma (\gamma + 1) \rho_0 \langle \eta_i^2 \rangle \frac{\partial}{\partial \eta} - 1\]  \hspace{1cm} (43)

(Ulmschneider, 1970, Bray and Loughhead, 1974). Hence, the wave pressure force \(f_w\) is of third order in the shock strength \(\eta\).

For the third term on the l.h.s. of Eq. (29) we obtain

\[f_{\text{sh}} = \frac{1}{\gamma} \langle (\rho_2 - \rho_1) U \rangle = c_S \frac{1}{\gamma} \sum_{i=1}^{l} \eta_i \rho_0 c_S \frac{\partial}{\partial \eta} - 1\]  \hspace{1cm} (44)

We used \(U = c_S\) and \(\rho_{m,i}/\rho_0 = v_{m,i}/c_S\) which are valid for weak shocks. This equation shows that the wave pressure due to shocks comes from the momentum deposition per wave period of the full shock jump. Compared to the wave pressure force \(f_w\), which is of third order in the shock strength \(\eta\), the contribution \(f_{\text{sh}}\) of the shock pressure is of first order and thus much larger if the shocks are sufficiently weak. We therefore neglect \(f_w\) relative to \(f_{\text{sh}}\). Eq. (44) thus gives the leading order wave pressure force due to weak shock waves. The hydrostatic equation, modified by the inclusion of wave pressure can thus be written

\[\frac{dp_0}{dz} - \rho_0 c_S \frac{\partial}{\partial \eta} - 1\]  \hspace{1cm} (45)

where \(p_0\) is the unperturbed gas pressure. Eq. (45) together with the equation governing the evolution of the shock strength \(\eta\) (see Ulmschneider, 1970), and a suitable energy equation, should allow to compute time-independent acoustically heated stellar atmosphere models.

As a simple example let us consider an isothermal atmosphere. The variation of the shock strength \(\eta\) for weak shock wave trains is given by

\[\frac{d \eta}{dz} = -\frac{1}{2} \frac{d}{dz} \left[ \frac{\rho_0 c_S^2}{\gamma} + \frac{1}{\rho_0 c_S^2} \frac{d}{dz} \left( \frac{1}{\gamma} \frac{d}{dz} \right) \right].\]  \hspace{1cm} (37)

(Ulmschneider, 1970; Bray and Loughhead, 1974). The pressure gradient can be eliminated by means of the hydrostatic equation. Since \(F_{\text{mech}}\) has to be calculated from the true hydrodynamic quantities and not from the time averaged ones, we have to use (45) without the correction term \(\rho_0 c_S \eta / \gamma\). It follows then, for constant \(c_S\) and \(\gamma\), that

\[\frac{d \eta}{dz} = \frac{\gamma}{c_S^2} \left( \frac{\gamma + 1}{\gamma c_S^2} \right) \]  \hspace{1cm} (46)

This equation is solved by

\[\eta(z) = \eta(z_0) \exp \left( \frac{z - z_0}{2H_0} \right)\]  \hspace{1cm} (48)

where \(H_0 = c_S^2 / (\gamma g)\) is the local hydrostatic pressure scale height (not including the effects of shocks), \(z_0\) the initial z and

\[\eta_\infty = g h \]  \hspace{1cm} (49)

is the limiting shock strength. The modified hydrostatic equation (45) then can be written as

\[\frac{d p_0}{dz} = -\rho_0 g \left[ 1 - \frac{\gamma - 1}{\gamma + 1} \eta_\infty \right].\]  \hspace{1cm} (50)
Hence, the momentum transfer of the shocks to the gas increases the local scale height of the atmosphere. The modified local scale height is

$$H = \frac{c_s^2}{\gamma g} \left[ 1 - \frac{\gamma}{\gamma + 1} \frac{\eta}{\eta_{\infty}} \right]^{-1}$$  \hspace{1cm} (51)

which approaches

$$H = H_0 (\gamma + 1)$$  \hspace{1cm} (52)

in the limit $\eta \to \eta_{\infty}$, i.e., within a few scale heights.

### 3.2. Larger amplitude shock waves

We now assume that the hydrodynamic shock waves are not necessarily weak, but that $u_2 < c_s$. The strength of the shock is given by the shock Mach number $M_s$,

$$M_s = \frac{U - u_1}{c_1},$$  \hspace{1cm} (53)

where $U$, $u_1$, and $c_1$ are the shock speed, the gas velocity, and the sound speed in front of the shock, respectively. For a perfect gas, the pressures $p_1$ and $p_2$ and the densities $\rho_1$ and $\rho_2$ in front and behind the shock are related by

$$\frac{p_2}{p_1} = \frac{\Theta}{\Theta - 1} \frac{\rho_2}{\rho_1} = \frac{(\gamma + 1) M_s^2}{(\gamma - 1) M_s^2 + 2}$$  \hspace{1cm} (54)

and

$$\frac{u_1 - U}{u_2 - U} = \frac{\rho_2}{\rho_1} = \frac{(\gamma + 1) M_s^2}{(\gamma - 1) M_s^2 + 2}$$  \hspace{1cm} (55)

(Landau and Lifshitz, 1959; note the correction of a misprint).

We now assume for the shock wave a similar velocity structure as in Eqs. (30) to (32). We obtain from Eqs. (53) and (55) for the velocity amplitudes $v_m$

$$v_m = \frac{u_2 - u_1}{2} = \frac{\Theta - 1}{2\Theta} M_s c_1.$$  \hspace{1cm} (56)

For an average sound speed $c_s^2 = (c_1^2 + c_2^2)/2$, with $c_2/c_1 = p_2/p_1/(\rho_1 \rho_2)$ and using Eqs. (54), (55) one has

$$c_1 = \sqrt{\frac{2}{\Theta + 1} c_0}$$  \hspace{1cm} (57)

which with Eq. (56) yields

$$v_m = \frac{\Theta - 1}{2\Theta} M_s \sqrt{\frac{2}{\Theta + 1} c_0}.$$  \hspace{1cm} (59)

Here $c_0$ can roughly be identified with the undisturbed sound speed. The wave pressure force $f_w$ due to the second term in Eq. (29) is the same as in Eq. (40), so that

$$f_w = \frac{d}{dz} \frac{1}{3} \rho_0 v_m^2,$$  \hspace{1cm} (60)

except now $v_m$ is now given by Eq. (59).

For the third term in Eq. (29) we find a shock pressure force of

$$f_{sh} = \frac{(\rho_2 - \rho_1)}{\rho} U,$$  \hspace{1cm} (61)

where from Eqs. (53) and (56) we have with $u_1 = -v_m$, valid for shocks with moderate strength, so

$$U = \frac{\Theta + 1}{2\Theta} M_s c_1,$$  \hspace{1cm} (62)

and with Eq. (57), we can write

$$U = \frac{\Theta - 1}{\Theta + 1} \rho_0 = \frac{M_s}{\Theta + 1} \sqrt{\frac{2}{\Theta + 1}} c_0.$$  \hspace{1cm} (63)

From Eq. (33) and with $\rho_0 = (\rho_2 + \rho_1)/2$ we find

$$\rho_2 - \rho_1 = \frac{2 M_s^2 - 2}{\gamma M_s^2 + 1} \rho_0,$$  \hspace{1cm} (64)

where $\rho_0$ is again roughly identified with the undisturbed density. The shock pressure force $f_{sh}$ is thus given by

$$f_{sh} = \frac{\rho_0 c_0}{\rho} \frac{2 M_s^2 - 2}{\gamma M_s^2 + 1} M_s \sqrt{\frac{2}{\Theta + 1}} \frac{2}{\Theta + 1} c_0.$$  \hspace{1cm} (65)

Expanding Eqs. (53), (55), (59) and (64) the weak shock results can be reproduced. With these result for $f_w$ and $f_{sh}$ the hydrostatic equilibrium equation modified by wave pressure can be written

$$\frac{d p}{dz} = \frac{d \rho_0}{dz} + f_w - f_{sh} = -\rho_0 g.$$  \hspace{1cm} (66)

### 4. Conclusions

We have derived expressions for the wave pressure in atmospheres permeated by trains of sawtooth-type shock waves both for weak shocks and somewhat stronger shocks. Two contributions modifying the hydrostatic equilibrium were found. The first contribution, $f_w$, arises from the nonlinear term in the momentum conservation equation and is present even in the absence of shocks. Due to the linear sawtooth shape of the waves considered by us, its numerical value is somewhat different from the form derived for sinusoidal waves. However, this wave pressure term, $f_w$, is small, and in the presence of shocks, can safely be neglected. The second contribution to the wave pressure, $f_{sh}$, arises from the transfer of the wave momentum to the gas at the shock discontinuity. The advantage of the present expressions of the wave pressure is that they allow to compute the behaviour of atmospheres permeated by a spectrum of shock waves of very different wavelength. Here short period waves could be treated with the present formalism, and long period waves with a time-dependent method without the need of a very fine mesh size.

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**References**


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