Astron. Astrophys. 234, 359–365 (1990)

ASTRONOMY AND ASTROPHYSICS

Wave pressure in stellar atmospheres due to shock wave trains

H.-P. Gail[1], **M. Cuntz**[1,2], **and P. Ulmschneider**[1]

[1] Institut für Theoretische Astrophysik der Universität Heidelberg, Im Neuenheimer Feld 561, D–6900 Heidelberg, Federal Republic of Germany

[2] Joint Institute for Laboratory Astrophysics and National Institute for Standards and Technology, University of Colorado, Boulder, CO 80309-0440, USA

Received December 6, 1989; accepted March 1, 1990

Abstract. In the present work we derive analytic expressions for the wave pressure of propagating shock wave trains in stellar atmospheres or winds. Applications to weak shocks and stronger shocks with sawtooth profiles are discussed in detail. The shocks are treated as discontinuities. Our results provide insight in the momentum balance of time-dependent stellar winds flows. The analytic expressions can be used as an independent test of hydrodynamic codes.

Key words: Stellar atmospheres – chromospheres – shocks – late-type stars

1. Introduction

Wave pressure is self–consistently included in time-dependent wave computations when the hydrodynamic equations are solved and the shocks are treated explicitly. The need to consider wave pressure separately arises only, when the momentum equation is solved in time–*independent* computations of stellar atmosphere or wind models. Recent time–dependent calculations of acoustically heated outer atmosphere models of late–type stars (see e.g. Ulmschneider, 1989) have again pointed out the well known *limiting shock strength* behaviour of monochromatic waves in gravitational atmospheres. This behaviour of shocks to reach a limiting strength arises from the balance of two opposing tendencies: Due to wave energy conservation, the rapid density decay in the stellar atmosphere leads to an amplitude growth, while on the other hand, shock dissipation results in a decrease of the wave amplitude. The validity of the limiting shock strength behaviour in wave computations where hydrogen ionization, radiation damping, shock dissipation and variable gravity are taken into account has been discussed by Cuntz and Ulmschneider (1988).

In monochromatic and stochastic short period wave models for the outer atmosphere of Aldebaran, Cuntz (1990) has recently investigated the influence of wave pressure on time-dependent stellar atmospheric structures. By time-averaging he evaluated the local actual pressure scale height and the local thermal pressure scale height in the models. The actual pressure scale height includes the full influence of the waves on the atmospheric structure while the thermal pressure scale height incorporates just the change of temperature and mean molecular weight. Cuntz found that the two scale heights are quite similar for monochromatic waves with small amplitudes. For stochastic waves, however, the actual pressure scale height becomes significantly larger than the thermal pressure scale height demonstrating the importance of the wave pressure by strong shocks.

The limiting strength property of the waves produces wave trains with rather typical and universal shape (see e.g. Ulmschneider, 1989) which is independent of the source of the acoustic energy generation. This opens the possibility to construct *time-independent* shock heated atmospheric models. The waves change the atmospheric model structure through their contribution in the momentum and energy equations. The momentum contribution is the wave pressure and the energy contribution is shock dissipation and radiation damping. For the energy contribution, heating formulae are already available in the literature (see e.g. Ulmschneider, 1970; Bray and Loughhead, 1974). Time-independent wave models can be calculated if both contributions are considered consistently. Such models give general insight in the behaviour of the waves and can easily be used as starting models for time-dependent computations.

There are older time-independent atmospheric model computations (e.g. McWhirter et al., 1975) which have used a wave pressure term, but due to an inadequate expression (propagation speed taken equal to the sound speed) for this term difficulties were encountered. Hartmann and McGregor (1980) have considered wave pressure in their Alfvén wave driven wind models computing the momentum transfer of the waves from their energy deposition. Further attempts to include wave pressure in time-independent wave models were discussed by Holzer and McGregor (1985) for various heating mechanisms. In the present work we want to obtain useful expressions for the wave pressure from shock wave trains. For this purpose, we derive, in Sect. 2, a general expression for the average momentum deposition by the shocks. In Sect. 3 we quantify this expression for weak shocks and larger amplitude shocks with sawtooth profiles. Section 4 gives our conclusions.

Send offprint requests to: H.-P. Gail

2. Momentum transfer of shock waves

We consider acoustic shock waves which travel through a stellar atmosphere. If there is only one shock (or at most a few shocks) present at any one time, it is most natural to consider the shock propagation through the atmosphere as a time-dependent problem. If, on the other hand, at any one instant, there exists a large number of shocks within the atmosphere, it will often be sufficient to consider only the average response of the atmosphere on the presence of an ensemble of shocks. In the following, we derive an equation for the average structure of such an atmosphere by taking the time-average of the momentum balance.

2.1. *Time average of the total momentum*

Let us consider some arbitrary but *fixed* volume element V within the stellar atmosphere. The total momentum of the gas contained in V is given by

$$P_\mu = \int_V dV\, \rho\, v_\mu . \tag{1}$$

Here the index μ indicates the different vector components. If we observe this volume element for a sufficiently long time interval T, many shocks traverse V. The momentum P_μ of the gas contained at each instant in V varies strongly with time and so does the time derivative $\partial P_\mu/\partial t$ which equals the instaneous net force excerted on the gas. The *average* net force excerted during the period T on the gas then is given by

$$\langle \dot{P}_\mu \rangle = \frac{1}{T} \int_0^T dt\, \frac{\partial P_\mu}{\partial t} . \tag{2}$$

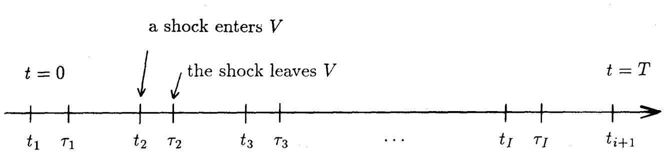

Fig. 1. Definition of the instants t_i and τ_i when a shock enters or leaves V, respectively.

For calculating this average, we assume V to be small enough that there is at most one shock in V. We denote by t_i the instants, at which a shock enters into V and by τ_i the instants, at which a shock leaves V (cf. Fig. 1). The index i enumerates the shocks traversing V during the period T. The total number of shocks crossing V during the time interval $[0, T]$ is denoted by I. We choose $[0, T]$ such that we have $t_1 = 0$ and $t_{I+1} = T$, i.e., we start the averaging process at the instant where a shock enters into V and finish it at the instant where the $(I+1)$–th shock enters V.

We have to consider two different cases:

1.) There is no shock in V. This holds during the I intervals of time $[\tau_i, t_{i+1}]$.

2.) There is one shock in V. This holds during the I intervals of time $[t_i, \tau_i]$.

These cases have to be treated differently in calculating the average momentum transfer.

In the first case during the time intervals between shocks we have

$$\frac{1}{T} \int_{\tau_i}^{t_{i+1}} dt\, \dot{P}_\mu = \frac{1}{T} \int_{\tau_i}^{t_{i+1}} dt \int_V dV\, \frac{\partial \rho v_\mu}{\partial t} . \tag{3}$$

Since no shocks are present, all hydrodynamic quantities are continuously differentiable and there holds the usual equation of momentum conservation

$$\frac{\partial \rho v_\mu}{\partial t} + \frac{\partial}{\partial x_\nu} \left[\rho v_\nu v_\mu + p \delta_{\mu\nu} \right] = -g_\mu \rho \tag{4}$$

where g_μ is the inwardly directed gravitational acceleration. Note that as costumary we assume summation if the same component index occurs twice in an expression. Inserting this into Eq. (3), observing that V is a fixed volume and interchanging integrations and differentiaton (permitted, since all quantities are continuously differentiable) yields

$$\frac{1}{T} \int_{\tau_i}^{t_{i+1}} dt\, \frac{\partial P_\mu}{\partial t} =$$

$$-\frac{1}{T} \int_V dV \left\{ \frac{\partial}{\partial x_\nu} \int_{\tau_i}^{t_{i+1}} dt \left[\rho v_\nu v_\mu + p \delta_{\mu\nu} \right] + g_\mu \int_{\tau_i}^{t_{i+1}} dt\, \rho \right\} . \tag{5}$$

On the other hand, we have

$$\frac{1}{T} \int_{\tau_i}^{t_{i+1}} dt\, \dot{P}_\mu = \frac{1}{T} \left[P_\mu(t_{i+1}) - P_\mu(\tau_i) \right] . \tag{6}$$

Hence it follows for the intervals $[\tau_i, t_{i+1}]$ between shocks

$$\frac{1}{T} \left[P_\mu(t_{i+1}) - P_\mu(\tau_i) \right] =$$

$$-\frac{1}{T} \int_V dV \left\{ \frac{\partial}{\partial x_\nu} \int_{\tau_i}^{t_{i+1}} dt \left[\rho v_\nu v_\mu + p \delta_{\mu\nu} \right] + g_\mu \int_{\tau_i}^{t_{i+1}} dt\, \rho \right\} . \tag{7}$$

In the second case, for all $t \in [t_i, \tau_i]$ there is one shock in V. We denote all quantities ahead of the shock by an index 1 and all quantities behind the shock by an index 2 (cf. Fig. 2). The shock can be considered as an infinitely thin layer in V, dividing the volume V in a pre–shock volume V_1 and a post–shock volume V_2. Since the jumps of the hydrodynamic quantities across the shock are bounded, any contribution of a spatial integration over the infinitely thin shock layer to the volume integration over V can be neglected. Then

$$P_\mu = \int_{V_2(t)} dV\, (\rho\, v_\mu)_2 + \int_{V_1(t)} dV\, (\rho\, v_\mu)_1 \tag{8}$$

with

$$V_1(t) + V_2(t) = V . \tag{9}$$

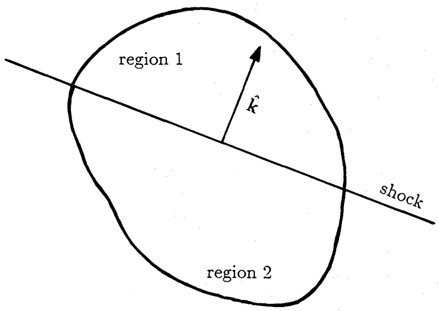

Fig. 2. Definition of the pre-shock region 1, the post-shock region 2 and the vector $\hat{\boldsymbol{k}}$

The volumes $V_1(t)$ and $V_2(t)$ depend on the time t since the shock moves across V. According to our definitions we have

$$\begin{aligned} V_1(t_i) = V \quad &, \quad V_2(t_i) = 0 \\ V_1(\tau_i) = 0 \quad &, \quad V_2(\tau_i) = V \end{aligned} \tag{10}$$

and

$$\begin{aligned} P_\mu(t_i) &= \int_V dV\, (\rho v_\mu)_1 \\ P_\mu(\tau_i) &= \int_V dV\, (\rho v_\mu)_2\,. \end{aligned} \tag{11}$$

$P_\mu(t)$ is a continuously differentiable function of the time $t \in]t_i, \tau_i[$, since this holds for $V_1(t)$ and $V_2(t)$ as well as for the integrands within V_1 and V_2. Thus, it follows

$$\frac{1}{T} \int_{t_i}^{\tau_i} dt\, \dot{P}_\mu = \frac{1}{T} \left[P_\mu(\tau_i) - P_\mu(t_i) \right] \tag{12}$$

and from Eqs. (11) we obtain for the intervals $[t_i, \tau_i]$ where shocks are present in V:

$$\frac{1}{T} \int_{t_i}^{\tau_i} dt\, \dot{P}_\mu = \frac{1}{T} \int_V dV \left[\left(\rho v_\mu(\tau_i)\right)_2 - \left(\rho v_\mu(t_i)\right)_1 \right]. \tag{13}$$

Note that the spatial integration is over the post–shock momentum at instant τ_i and over the pre-shock momentum at instant t_i, respectively.

2.2. Average momentum change

By combining Eqs. (6) and (12) for all I intervals with and without shocks, we obtain

$$\langle \dot{P}_\mu \rangle = \frac{1}{T} \int_0^T dt\, \dot{P}_\mu = \frac{1}{T} \left[P_\mu(T) - P_\mu(0) \right] \tag{14}$$

since the contributions of all division points t_i and τ_i of the t–axis between $t = 0$ and $t = T$ cancel pairwise. On the other hand, by combining Eqs. (7) and (13) for all our intervals, we obtain by means of (14)

$$\begin{aligned} &\frac{1}{T} \left[P_\mu(T) - P_\mu(0) \right] = \\ &-\frac{1}{T} \int_V dV \left\{ \frac{\partial}{\partial x_\nu} \sum_{i=1}^{I} \int_{\tau_i}^{t_{i+1}} dt \left[\rho v_\nu v_\mu + p \delta_{\mu\nu} \right] + g_\mu \sum_{i=1}^{I} \int_{\tau_i}^{t_{i+1}} dt\, \rho \right\} \\ &+ \frac{1}{T} \int_V dV \sum_{i=1}^{I} \left[\left(\rho v_\mu(\tau_i)\right)_2 - \left(\rho v_\mu(t_i)\right)_1 \right]. \end{aligned} \tag{15}$$

We divide this equation by the volume V and take the limit of an infinitely small volume element V. In this limit the volume integrals divided by V tend to the value of the integrands at the place of the infinitely small volume element and simultaneously the instants τ_i become identical with the instants t_i. The term in brackets in the second term in Eq. (15) becomes equal to the jump $[(\rho v_\mu)_{2,i} - (\rho v_\mu)_{1,i}]$ of ρv_μ across the shock front. It follows

$$\begin{aligned} &\frac{1}{T} \left[\rho v_\mu \Big|_{t=T} - \rho v_\mu \Big|_{t=0} \right] = -\frac{\partial}{\partial x_\nu} \left[\langle \rho v_\mu v_\nu \rangle + \langle p \rangle \delta_{\mu\nu} \right] - g_\mu \langle \rho \rangle \\ &+ \frac{1}{T} \sum_{i=1}^{I} \left[(\rho v_\mu)_{2,i} - (\rho v_\mu)_{1,i} \right] \end{aligned} \tag{16}$$

where

$$\langle A \rangle = \frac{1}{T} \sum_{i=1}^{I} \int_{t_{i-1}}^{t_i} dt\, A(t) \tag{17}$$

is the time average of some quantity A between shocks.

The last term on the r.h.s. of Eq. (16) is the average of the momentum jump ρv_μ across the shock front. This momentum is transferred to the gas. Let $\hat{\boldsymbol{k}}_i$ be the normal of the i–th shock front. Then

$$(\rho v_\mu)_{2,i} - (\rho v_\mu)_{1,i} = \left[(\rho v)_{2,i} - (\rho v)_{1,i} \right] \hat{k}_{\mu,i} \tag{18}$$

where v is the velocity component of the gas in the direction of $\hat{\boldsymbol{k}}_i$. We denote the velocity of the $i-th$ shock front in the stellar frame of rest by U_i and the pre-shock and post-shock velocity components of the gas normal to the shock front in the frame where the shock front is at rest by $u_{1,i}$ and $u_{i,2}$, respectively. Then

$$v_{1,i} = u_{1,i} + U_i, \qquad v_{2,i} = u_{2,i} + U_i, \tag{19}$$

and from the Hugoniot condition,

$$(\rho u)_{1,i} = (\rho u)_{2,i}\,, \tag{20}$$

of mass conservation across the shock we obtain

$$(\rho v_\mu)_{2,i} - (\rho v_\mu)_{1,i} = \left[\rho_{2,i} - \rho_{1,i} \right] U_i\, \hat{k}_{\mu,i}\,. \tag{21}$$

We define the mean time between shocks by

$$\mathscr{P} = \frac{T}{I} \tag{22}$$

and the average momentum transfer of a shock to the gas by

$$\frac{1}{I} \sum_{i=1}^{I} (\rho_{2,i} - \rho_{1,i})\, U_i\, \hat{k}_{\mu,i} = \langle (\rho_2 - \rho_1) U_i \hat{k}_\mu \rangle\,. \tag{23}$$

Further, we note that for a sufficiently long time T the l.h.s of Eq. (16) becomes negligibly small if the pre–shock momenta ρv_μ at $t = 0$ and $t = T$ on the average vary slower than linear with T, which usually will be the case in a stationary atmosphere. Then it follows that

$$\frac{\partial}{\partial x_\nu}\left[\langle p\rangle \delta_{\mu\nu} + \langle \rho v_\mu v_\nu\rangle\right] - \frac{1}{\mathscr{P}}\langle(\rho_2 - \rho_1)U\hat{k}_\mu\rangle = -g_\mu\langle\rho\rangle\,. \tag{24}$$

This is our final result for the average momentum balance in an atmosphere with many shocks. The second and third terms on the l.h.s. of Eq. (24) represent the wave pressure. The second term, which is always different from zero in an atmosphere with non–vanishing fluid motions, is present even in absence of shocks. This represents the usual wave pressure of an ensemble of sound waves. The third term on the l.h.s. represents the average momentum transfer of shocks to the gas which enters into the equation like an external force. Its form can easily be understood by the following consideration assuming periodic shock waves. Let u_1 and u_2 be the velocity in front of and behind a shock of speed U. Then $\rho_2 u_2 - \rho_1 u_1 = (\rho_2 - \rho_1)U$ is the change in momentum density at a fixed position following shock passage. If the interval in time between successive shocks is $\mathscr{P}$, then with the passage of each shock, a parcel of gas experiences an impulsive force $(\rho_2 - \rho_1)U/\mathscr{P}$.

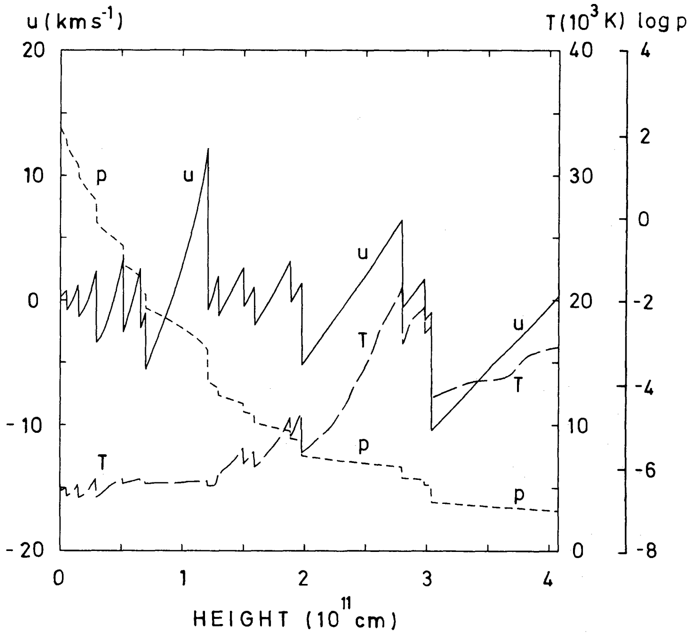

Fig. 3. Snapshot of a time-dependent stochastic wave calculation for Arcturus. The run of the flow speed u, the sound speed c, the temperature T and the pressure p are shown.

2.3. *Plane parallel atmospheres*

With fluid motions and shocks present in an atmosphere, we cannot expect that there exists any symmetry for the actual hydrodynamic variables. The time averaged quantities in the absence of strong magnetic fields and systematic fluid motions, however, should show the same kind of symmetry as the local gravitational field, i.e., invariance with respect to rotations about an arbitrary axis parallel to the direction of the vector g_μ. The most general second rank tensor invariant with respect to this symmetry operation is a linear combination of the two fundamental tensors $\delta_{\mu\nu}$ and $\hat{n}_\mu\hat{n}_\nu$, with $\hat{\boldsymbol{n}}$ being the unit vector parallel to g_μ. Thus

$$\langle\rho v_\mu v_\nu\rangle = a(z)\,\delta_{\mu\nu} + b(z)\,\hat{n}_\mu\hat{n}_\nu\,. \tag{25}$$

a and b may depend on the coordinate z parallel to g_μ. By contracting Eq.(25) and by multiplying Eq.(25) with $\hat{n}_\mu\hat{n}_\nu$ one easily shows

$$\langle\rho v_\mu v_\nu\rangle = \frac{1}{2}\langle\rho v_\perp^2\rangle\,\delta_{\mu\nu} + \left\{\langle\rho v_\parallel^2\rangle - \frac{1}{2}\langle\rho v_\perp^2\rangle\right\}\,\hat{n}_\mu\hat{n}_\nu \tag{26}$$

where $v_\perp$ and $v_\parallel$ are the velocity components perpendicular and parallel to $\hat{\boldsymbol{n}}$, respectively. Then, since the averages $\langle\rho v_\perp^2\rangle$ and $\langle\rho v_\parallel^2\rangle$ depend only on the z-coordinate,

$$\frac{\partial}{\partial x_\nu}\langle\rho v_\mu v_\nu\rangle = \hat{n}_\mu\frac{\partial\,\langle\rho v_\parallel^2\rangle}{\partial z}\,. \tag{27}$$

For the same symmetry reasons, the average momentum transfer of shocks is a vector parallel to g_μ, thus

$$\langle(\rho_2 - \rho_1)U\hat{\boldsymbol{k}}\rangle = \langle(\rho_2 - \rho_1)U\hat{\boldsymbol{k}}\cdot\hat{\boldsymbol{n}}\rangle\,\hat{\boldsymbol{n}}\,. \tag{28}$$

Combining Eqs. (24) to (28) yields

$$\frac{\partial\,\langle p\rangle}{\partial z} + \frac{\partial\,\langle\rho v_\parallel^2\rangle}{\partial z} - \frac{1}{\mathscr{P}}\langle(\rho_2 - \rho_1)U\hat{\boldsymbol{k}}\cdot\hat{\boldsymbol{n}}\rangle = -g\langle\rho\rangle\,, \tag{29}$$

which is valid for the average momentum balance in a plane parallel stellar atmosphere.

3. Wave pressure for different shock trains

In this section, we derive expressions for the wave pressure for the case of vertically propagating shock wave trains. A typical example of a shocked atmosphere is shown in Fig. 3 (Cuntz, 1987). This result of a numerical solution of the full system of hydrodynamic equations demonstrates clearly that the variation of the hydrodynamic quantities between shocks can rather accurately be approximated by linear segments (except, perhaps, for the very strongest shocks). For this reason, in the following, we restrict our considerations to sawtooth shock waves. We first consider weak shocks and then somewhat larger amplitude shocks, where the velocity amplitude is still considerably smaller than the sound speed. The time-averaged atmosphere is assumed to be at rest.

3.1. *Weak shocks*

We consider an ensemble of weak shocks propagating outwards parallel to g_μ in a plane-parallel stellar atmosphere. For small amplitude sawtooth waves one has the following velocity, density and pressure variations at a fixed point within the atmosphere

$$v(t) = v_{m,i} - 2v_{m,i}\frac{t}{\mathscr{P}_i}\,, \tag{30}$$

$$\rho(t) = \rho_0 + \rho_{m,i} - 2\rho_{m,i}\frac{t}{\mathscr{P}_i}, \tag{31}$$

$$p(t) = p_0 + p_{m,i} - 2p_{m,i}\frac{t}{\mathscr{P}_i}, \tag{32}$$

where $\mathscr{P}_i = t_i - t_{i-1}$ is the time between passage of the $(i-1)-th$ and $i-th$ shocks, subscript m indicates maximum amplitudes, and ρ_0 and p_0 are the unperturbed density and pressure, respectively. We assume that $v_0 = 0$, i.e., that no average flow occurs in the atmosphere. Then one has

$$\langle p \rangle = \frac{1}{T}\sum_{i=1}^{I}\int_0^{\mathscr{P}_i} dt\, p(t) = \frac{1}{T}\sum_{i=1}^{I} p_0 \mathscr{P}_i = p_0 \tag{33}$$

since $\sum_i \mathscr{P}_i = T$ and correspondingly

$$\langle \rho \rangle = \rho_0 . \tag{34}$$

For the second term at the l.h.s. of Eq. (29) we obtain

$$\langle \rho v_\parallel^2 \rangle = \frac{1}{T}\sum_{i=1}^{I}\int_0^{\mathscr{P}_i} dt\, \rho(t)\, v^2(t) = \frac{1}{3}\rho_0 \sum_{i=1}^{I} v_{m,i}^2 \frac{\mathscr{P}_i}{T}. \tag{35}$$

For weak shocks one has for the velocity jump across the shock front

$$v_{m,i} = \frac{1}{2} c_S \eta_i \tag{36}$$

where

$$\eta_i = \frac{\rho_2 - \rho_1}{\rho_1} \ll 1 \tag{37}$$

is the shock strength and c_S the sound speed (Ulmschneider, 1970). ρ_1, ρ_2 are the densities in front and behind the shock. From Eqs. (35) to (37) it follows that

$$\langle \rho v_\parallel^2 \rangle = \frac{1}{12}\rho_0 c_S^2 \langle \eta^2 \rangle, \tag{38}$$

where

$$\langle \eta^2 \rangle = \sum_{i=1}^{I} \eta_i^2 \frac{\mathscr{P}_i}{T} \tag{39}$$

is the square average of the shock strength. The second term at the l.h.s. of Eq. (29) can then be written as a wave pressure force

$$f_w = \frac{d}{dz}\frac{1}{12}\rho_0\, c_S^2\, \langle \eta^2 \rangle \tag{40}$$

or, alternatively, as

$$f_w = \frac{d}{dz}\frac{F_{mech}}{c_S}, \tag{41}$$

since $\gamma p_0 = c_S^2 \rho_0$ and

$$F_{mech} = \frac{1}{12}\gamma p_0 c_S \langle \eta^2 \rangle . \tag{42}$$

Eq.(42) represents the mechanical energy flux of sawtooth shock waves (Ulmschneider, 1970, note difference of notation of F_{mech}). For weak shocks, the dissipation of F_{mech} with height is given by

$$\frac{dF_{mech}}{dz} = -\frac{1}{12}\gamma(\gamma+1)p_0\langle \eta^3 \rangle\, \mathscr{P}^{-1} \tag{43}$$

(Ulmschneider, 1970, Bray and Loughhead, 1974). Hence, the wave pressure force f_w is of third order in the shock strength η.

For the third term on the l.h.s. of Eq. (29) we obtain

$$f_{sh} = \frac{1}{\mathscr{P}}\langle(\rho_2 - \rho_1)U\rangle = \frac{c_S}{\mathscr{P}}\frac{1}{I}\sum_{i=1}^{I}\eta_i \rho_{1,i} \approx \frac{\rho_0 c_S}{\mathscr{P}}\langle \eta \rangle . \tag{44}$$

We used $U = c_S$ and $\rho_m/\rho_0 = v_m/c_S$ which are valid for weak shocks. This equation shows that the wave pressure due to shocks comes from the momentum deposition per wave period of the full shock jump. Compared to the wave pressure force f_w, which is of third order in the shock strength η, the contribution f_{sh} of the shock pressure is of first order and thus much larger if the shocks are sufficiently weak. We therefore neglect f_w relative to f_{sh}. Eq. (44) thus gives the leading order wave pressure force due to weak shock waves. The hydrostatic equation, modified by the inclusion of wave pressure can thus be written

$$\frac{dp_0}{dz} - \frac{\rho_0 c_S}{\mathscr{P}}\eta = -\rho_0 g\ , \tag{45}$$

where p_0 is the unperturbed gas pressure. Eq. (45) together with the equation governing the evolution of the shock strength η (see Ulmschneider, 1970), and a suitable energy equation, should allow to compute time-independent acoustically heated stellar atmosphere models.

As a simple example let us consider an isothermal atmosphere. The variation of the shock strength η for weak shock wave trains is given by

$$\frac{d\eta}{dz} = -\frac{\eta}{2}\left[\frac{1}{p_0}\frac{dp_0}{dz} + \frac{(\gamma+1)\eta}{c_S\mathscr{P}} + \frac{1}{\gamma}\frac{d\gamma}{dz} + \frac{3}{2}\frac{1}{c_S^2}\frac{dc_S^2}{dz}\right] \tag{46}$$

(Ulmschneider, 1970; Bray and Loughhead, 1974). The pressure gradient can be eliminated by means of the hydrostatic equation. Since F_{mech} has to be calculated from the true hydrodynamic quantities and not from the time averaged ones, we have to use (45) without the correction term $\rho_0 c_S \eta/\mathscr{P}$. It follows then, for constant c_S and γ, that

$$\frac{d\eta}{dz} = \frac{\eta}{2}\left[\frac{\gamma}{c_S^2}g - \frac{(\gamma+1)\eta}{c_S\mathscr{P}}\right] . \tag{47}$$

This equation is solved by

$$\eta(z) = \eta(z_\ell)\exp\left(\frac{z - z_\ell}{2H_0}\right)\cdot \left[1 + \frac{\eta(z_\ell)}{\eta_\infty}\left(\exp\left(\frac{z - z_\ell}{2H_0}\right) - 1\right)\right]^{-1} . \tag{48}$$

Here $H_0 = c_S^2/(\gamma g)$ is the local hydrostatic pressure scale height (not including the effects of shocks), z_ℓ the initial z and

$$\eta_\infty = \frac{\gamma g \mathscr{P}}{(\gamma+1)c_S} \tag{49}$$

is the *limiting shock strength*. The modified hydrostatic equation (45) then can be written as

$$\frac{dp_0}{dz} = -\rho_0 g\left[1 - \frac{\gamma}{\gamma+1}\frac{\eta}{\eta_\infty}\right] . \tag{50}$$

Hence, the momentum transfer of the shocks to the gas increases the local scale height of the atmosphere. The modified local scale hight is

$$H = \frac{c_S^2}{\gamma g}\left[1 - \frac{\gamma}{\gamma+1}\frac{\eta}{\eta_\infty}\right]^{-1} \tag{51}$$

which approches

$$H = H_0(\gamma + 1) \tag{52}$$

in the limit $\eta \to \eta_\infty$, i.e., within a few scale heights.

3.2. Larger amplitude shock waves

We now assume that the hydrodynamic shock waves are not necessarily weak, but that $u_2 < c_S$. The strength of the shock is given by the shock Mach number M_S,

$$M_S \equiv \frac{U - u_1}{c_1}, \tag{53}$$

where U, u_1 and c_1 are the shock speed, the gas velocity and the sound speed in front of the shock, respectively. For a perfect gas the pressures p_1 and p_2 and the densities ρ_1 and ρ_2 in front and behind the shock are related by

$$\frac{p_2}{p_1} = \Phi = \frac{2\gamma M_S^2 - \gamma + 1}{\gamma + 1} \tag{54}$$

and

$$\frac{u_1 - U}{u_2 - U} = \frac{\rho_2}{\rho_1} = \Theta = \frac{(\gamma+1)M_S^2}{(\gamma-1)M_S^2 + 2} \tag{55}$$

(Landau and Lifshitz, 1959; note the correction of a misprint).

We now assume for the shock wave a similar velocity structure as in Eqs. (30) to (32). We obtain from Eqs. (53) and (55) for the velocity amplitude v_m

$$v_m = \frac{u_2 - u_1}{2} = \frac{\Theta - 1}{2\Theta} M_S\, c_1\,. \tag{56}$$

For an average sound speed $c_0^2 = (c_2^2 + c_1^2)/2$, with $c_2/c_1 = p_2\rho_1/(p_1\rho_2)$ and using Eqs. (54), (55) one has

$$c_1 = \sqrt{\frac{2}{\frac{\Phi}{\Theta} + 1}}\, c_0\,, \tag{57}$$

which with Eq. (56) yields

$$v_m = \frac{\Theta - 1}{2\,\Theta} M_S \sqrt{\frac{2}{\frac{\Phi}{\Theta} + 1}}\, c_0\,. \tag{59}$$

Here c_0 can roughly be identified with the undisturbed sound speed. The wave pressure force f_w due to the second term in Eq. (29) is the same as in Eq. (40), so that

$$f_w = \frac{d}{dz}\,\frac{1}{3}\rho_0 v_m^2\,, \tag{60}$$

except now v_m is now given by Eq. (59).

For the third term in Eq. (29) we find a shock pressure force of

$$f_{sh} = \frac{(\rho_2 - \rho_1)\, U}{P}\,, \tag{61}$$

where from Eqs. (53) and (56) we have with $u_1 = -v_m$, valid for shocks with moderate strength, so

$$U = \frac{\Theta + 1}{2\Theta} M_S\, c_1\,, \tag{62}$$

and with Eq. (57), we can write

$$U = 2\frac{\Theta - 1}{\Theta + 1}\rho_0 = \frac{M_S}{\Theta + 1}\sqrt{\frac{2}{\frac{\Phi}{\Theta} + 1}}\, c_0\,. \tag{63}$$

From Eq. (33) and with $\rho_0 = (\rho_2 + \rho_1)/2$ we find

$$\rho_2 - \rho_1 = \frac{2M_S^2 - 2}{\gamma M_S^2 + 1}\,\rho_0\,, \tag{64}$$

where ρ_0 is again roughly identified with the undisturbed density. The shock pressure force f_{sh} is thus given by

$$f_{sh} = \frac{\rho_0\, c_0}{\mathscr{P}}\,\frac{2M_S^2 - 2}{\gamma M_S^2 + 1}\,\frac{M_S}{\Theta + 1}\sqrt{\frac{2}{\frac{\Phi}{\Theta} + 1}}\,, \tag{65}$$

Expanding Eqs. (53), (55), (59) and (64) the weak shock results can be reproduced. With these result for f_w and f_{sh} the hydrostatic equilibrium equation modified by wave pressure can be written

$$\frac{dp}{dz} = \frac{dp_0}{dz} + f_w - f_{sh} = -\rho_0 g\,. \tag{66}$$

4. Conclusions

We have derived expressions for the wave pressure in atmospheres permeated by trains of sawtooth-type shock waves both for weak shocks and somewhat stronger shocks. Two contributions modifying the hydrostatic equilibrium were found. The first contribution, f_w, arises from the nonlinear term in the momentum conservation equation and is present even in the absence of shocks. Due to the linear sawtooth shape of the waves considered by us, its numerical value is somewhat different from the form derived for sinusoidal waves. However, this wave pressure term, f_w, is small, and in the presence of shocks, can safely be neglected. The second contribution to the wave pressure, f_{sh}, arises from the transfer of the wave momentum to the gas at the shock discontinuity. The advantage of the present expressions of the wave pressure is that they allow to compute the behaviour of atmospheres permeated by a spectrum of shock waves of very different wavelength. Here short period waves could be treated with the present formalism, and long period waves with a time-dependent method without the need of a very fine mesh size.

Acknowledgements. This work has been supported by the Deutsche Forschungsgemeinschaft (Projekt Cu19/1-1, Ul 57/11-2) and by the National Aeronautics and Space Administration (Grant NGL 06-003-057 to the University of Colorado)

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