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On the interaction of longitudinal and transversal waves in thin magnetic flux tubes

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Abstract. An analytical investigation of the nonlinear interaction of longitudinal and transversal waves in thin magnetic flux tubes is presented and the nonlinear terms which give rise to wave generation of other modes and to shock formation are isolated. The nonlinear resonant three-wave interaction of longitudinal and transversal waves is studied together with the growth and decay behaviour of these waves. This analytical study clarifies our previous numerical computations of nonlinear wave generation and of the steepening of longitudinal as well as transversal wave profiles.

Key words: magnetohydrodynamics – MHD waves – shock waves

1. Introduction

In recent years, particular attention has been given to various problems of MHD wave propagation in magnetic flux tubes. A main reason for this interest is the fact, that MHD waves may be one of the basic mechanisms both for the heating of the outer layers of late-type stellar atmospheres and for the production of stellar winds (see e.g. Narain & Ulmschneider 1990, 1995, Parker 1991; Hartmann & MacGregor 1982, Holtzer, Fla & Leer 1983, Cally 1987, Moore et al. 1991).

The fact, that the magnetic fields in the lower layers of the solar photosphere have a filamentary structure and are concentrated in thin flux tubes, brings special features to the MHD wave propagation, such as the appearance of new kinds of modes and of dispersion. Unfortunately, the gravity induced pressure and density stratification of stellar atmospheres leads to a flaring of the magnetic tubes and a strong growth of the wave amplitude which complicates a theoretical description of MHD wave propagation, especially in the nonlinear regime. A summary of general results of the linear theory of MHD waves in magnetic flux tubes may be found in the review of Roberts (1992).

A convenient model for the investigation of MHD waves in magnetic flux tubes is the *thin tube approximation* (Defouw 1976, Roberts & Webb 1978, 1979, Spruit 1981). The use of this approximation allows to decrease the number of physical variables and arguments. Its main idea is to consider the tube as a one-dimensional sequence of mass elements, where spatial scales are large with respect to the tube radius and where the physical variables do not change much across the tube from their values on the tube axis. There are two different approaches in the thin tube approximation. The first, suggested by Roberts & Webb (1978) (see also Ferriz-Mas & Schüssler 1989 and Ferriz-Mas, Schüssler & Anton 1989), allows to consider the dynamics of longitudinal (sausage) and torsional (Alfvén) modes in the straight vertical tube. Kinking motions of the tube is not considered in this model. The second approach was proposed by Spruit (1981). His model allows to consider nonsymmetrical perturbations of the tube (kink modes). The Spruit model has been criticized by Choudhuri (1990) and Cheng (1992). Cheng (1992) has rewritten the modified Spruit model in a very convenient intrinsic form. There are two types of propagating modes in the model, namely transversal (kink) and longitudinal (sausage) modes. These waves are non-dispersive, i.e. a dependence of the wave frequency upon the wave number is linear. Taking into account the wave structure in the tube surrounding may lead to an appearance of dispersion (Roberts 1992).

Modern observations show the presence of large amplitude wave motions in the magnetic flux tubes (see e.g. Muller et al. 1994, Roudier et al. 1994); consequently the wave dynamics will often be nonlinear. In the absence of dispersion, nonlinear processes lead to a formation of shock waves in the tube (see e.g. Herbold et al. 1985, Ferriz-Mas & Moreno-Insertis 1987, Zhugzhda, Bromm & Ulmschneider 1995). The latter au-

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thors have found that there is the possibility of longitudinal-transversal shock wave formation in the thin tube.

An elementary process, which is the physical basis of the shock wave formation, is the nonlinear generation of second harmonics of waves. Aside of this effect, the presence of several types of modes gives an additional mechanism for the shock wave formation, namely the nonlinear interaction of different types of waves. This nonlinear mode-coupling may be a reason for the production of longitudinal waves in higher layers of the solar atmosphere and this effect appears to be important for the problem of coronal heating. Ulmschneider, Zähringer & Musielak (1991) have numerically explored the nonlinear behaviour of transversal tube waves and the mode-coupling between transversal and longitudinal waves. These results indicate that nonlinear interactions are very important for these waves. It would be interesting to provide an analytical basis for these numerical results.

The aim of this work is to explore analytically the nonlinear mode-coupling of transversal and longitudinal waves in the thin magnetic flux tube. The work is based on Cheng's intrinsic form of the thin tube equations. We do not take into account effects connected with gravity and steady flows in the tube. We assume that the tube is vertical and straight. In Sect. 2 we show the approximations used. In Sect. 3 we discuss the linear propagation of MHD tube waves in the model. A set of reduced equations for the complex amplitudes of interacting waves is obtained in Sect. 4. Solutions of the set of equations are discussed in Sect. 5. A summary and our conclusions are given in Sect. 6.

2. Governing equations

Consider a thin straight, vertical magnetic flux tube in the absence of gravity and dissipation. The tube is assumed to rest in a nonmagnetic external medium. As a set of basic equations we use the self-consistent equations derived by Cheng (1992, 1994). The set is a modification of Spruit's equations (Spruit 1981), which takes into account the backreaction on the tube by the surrounding plasma flowing around the tube.

The tangential and transverse equations of motions are given by

$$\frac{\partial v_1}{\partial t} - v_2\frac{\partial \theta}{\partial t} + v_r\frac{\partial v_r}{\partial s} = -\frac{1}{\rho}\frac{\partial p}{\partial s}, \tag{1}$$

$$\frac{\partial v_2}{\partial t} + \left(v_1 + \frac{\rho-\rho_e}{\rho+\rho_e}v_r\right)\frac{\partial \theta}{\partial t} + \frac{\rho}{\rho+\rho_e}\left(v_r^2 - \frac{B^2}{4\pi\rho}\right)\frac{\partial \theta}{\partial s} = 0, \tag{2}$$

where ρ, p, v_1 and v_2 are density, pressure, tangential and transverse velocities of the fluid inside the tube, respectively. v_r is the velocity of the plasma relative to the tube and is defined as $v_r = (\partial s/\partial t)_{s_0}$, s being the arc-length and s_0 the arc-length at the initial time. Variable θ is the inclination of the tube with respect to a fixed direction in the plane. The external density of the plasma is ρ_e, B is the magnetic field strength. The adiabatic energy equation is given by

$$\frac{d}{dt}\left(\frac{p}{\rho^\gamma}\right) = 0, \tag{3}$$

where γ is the ratio of specific heats.

The equations of continuity and induction may be combined into one equation:

$$\frac{\partial}{\partial t}\frac{\rho}{B} + \frac{\partial}{\partial s}\frac{\rho v_r}{B} = 0. \tag{4}$$

The variables v_1, v_2, v_r and θ are connected by the following relations:

$$\frac{\partial(v_1 - v_r)}{\partial s} - v_2\frac{\partial \theta}{\partial s} = 0, \tag{5}$$

$$\frac{\partial v_2}{\partial s} + (v_1 - v_r)\frac{\partial \theta}{\partial s} = \frac{\partial \theta}{\partial t}. \tag{6}$$

We assume pressure balance of the tube and its surroundings:

$$p + B^2/8\pi = p_e, \tag{7}$$

where p and p_e are the internal and external gas pressures, respectively. The flows of the surrounding plasma are assumed to be potential (Spruit 1981). Equations (1)-(7) is a self- consistent hyperbolic system with two independent variables, time t and the arc-length s.

3. Small nonlinearity limit equations

Consider the dynamics of small perturbations of the plasma variables from the equilibrium state described by the unperturbed internal ρ_0 and external ρ_e densities, as well as the internal magnetic field B_0, the internal p_0 and external p_e gas pressures. There are no stationary flows in the system. From Eq. (7), the stable state variables are connected by the condition

$$p_0 + B_0^2/8\pi = p_e. \tag{8}$$

We look for the solution of the set (1)-(7) in the following form

$$\begin{aligned} &p = p_0 + \mu\tilde{p}, \quad \rho = \rho_0 + \mu\tilde{\rho}, \quad B = B_0 + \mu\tilde{B}, \\ &v_1 = \mu\tilde{v}_1, \quad v_2 = \mu\tilde{v}_2, \quad v_r = \mu\tilde{v}_r, \quad \theta = \mu\tilde{\theta}, \end{aligned} \tag{9}$$

where the dimensionless quantity μ is the small positive parameter of nonlinearity. The quantities with tilde are full perturbations, and include both linear and nonlinear disturbunces. For ease of notation the tilde will be omitted in the following development.

Substituting expansion (9) in Eqs. (1)-(7), and neglecting terms of the third and higher orders in μ, we obtain the following equations:

$$\frac{\partial v_1}{\partial t} + \frac{1}{\rho_0}\frac{\partial p}{\partial s} = \mu S_1, \tag{10}$$

$$\frac{\partial v_2}{\partial t} - C_A^2 \frac{\rho_0}{\rho_0 + \rho_e} \frac{\partial \theta}{\partial s} = \mu S_2, \tag{11}$$

$$\frac{\partial p}{\partial t} - C_s^2 \frac{\partial \rho}{\partial t} = \mu S_3, \tag{12}$$

$$\frac{\partial}{\partial t}\left(\frac{\rho}{\rho_0} - \frac{B}{B_0}\right) + \frac{\partial v_r}{\partial s} = \mu S_4, \tag{13}$$

$$\frac{\partial v_1}{\partial s} - \frac{\partial v_r}{\partial s} = \mu S_5, \tag{14}$$

$$\frac{\partial v_2}{\partial s} - \frac{\partial \theta}{\partial t} = \mu S_6, \tag{15}$$

$$p + \frac{B_0 B}{4\pi} = \mu S_7, \tag{16}$$

where $C_A = B_0/\sqrt{4\pi\rho_0}$ is the Alfvén speed, $C_s = \sqrt{\gamma p_0/\rho_0}$ is the sound speed. Eqs. (10)—(16) contain only terms of first ($\sim \mu$) and second ($\sim \mu^2$) orders (terms of third and higher orders in μ have been neglected). On the left hand side, the linear terms ($\sim \mu$) are gathered and on the right hand side the quadratic nonlinear terms ($\sim \mu^2$) are gathered with:

$$S_1 = v_2 \frac{\partial \theta}{\partial t} - v_r \frac{\partial v_r}{\partial s} + \frac{\rho}{\rho_0^2}\frac{\partial p}{\partial s}, \tag{17}$$

$$S_2 = -\left(v_1 + \frac{\rho_0 - \rho_e}{\rho_0 + \rho_e} v_r\right)\frac{\partial \theta}{\partial t} +$$

$$\frac{\rho_0}{\rho_0 + \rho_e} C_A^2 \left(\frac{2B}{B_0} - \frac{\rho}{\rho_0 + \rho_e}\right)\frac{\partial \theta}{\partial s}, \tag{18}$$

$$S_3 = v_r \frac{\partial}{\partial s}(C_s^2 \rho - p) + \frac{\gamma}{\rho_0}\frac{\partial}{\partial t}(p\rho) - C_s^2 \frac{\gamma+1}{2}\frac{1}{\rho_0}\frac{\partial \rho^2}{\partial t}, \tag{19}$$

$$S_4 = \frac{\partial}{\partial t}\left[\left(\frac{\rho}{\rho_0} - \frac{B}{B_0}\right)\frac{B}{B_0}\right] + \frac{\partial}{\partial s}\left[\left(\frac{B}{B_0} - \frac{\rho}{\rho_0}\right) v_r\right], \tag{20}$$

$$S_5 = v_2 \frac{\partial \theta}{\partial s}, \tag{21}$$

$$S_6 = (v_r - v_1)\frac{\partial \theta}{\partial s}, \tag{22}$$

$$S_7 = -B^2/8\pi. \tag{23}$$

Equations (10)-(16) may be combined into two second order differential equations:

$$\left(\frac{\partial^2}{\partial t^2} - C_k^2 \frac{\partial^2}{\partial s^2}\right) v_2 = \mu\left(\frac{\partial S_2}{\partial t} - C_k^2 \frac{\partial S_6}{\partial s}\right), \tag{24}$$

$$\left(\frac{\partial^2}{\partial t^2} - C_T^2 \frac{\partial^2}{\partial s^2}\right)\rho =$$

$$\mu \frac{\rho_0 C_A^2}{C_A^2 + C_s^2}\left\{\frac{\partial}{\partial t}\left[S_4 + S_5 + \frac{4\pi}{B_0^2}\left(\frac{\partial S_7}{\partial t} - S_3\right)\right] + \right.$$

$$\left.\frac{\partial}{\partial s}\left[\frac{1}{\rho_0}\int \frac{\partial S_3}{\partial s} dt - S_1\right]\right\}. \tag{25}$$

Here we use the usual notations of the *kink* speed

$$C_k = \sqrt{\frac{\rho_0}{\rho_0 + \rho_e}} C_A, \tag{26}$$

and the *tube* speed

$$C_T = \frac{C_A C_s}{\sqrt{C_A^2 + C_s^2}}. \tag{27}$$

Note, that the ratio of the internal unperturbed density ρ_0 and external unperturbed density ρ_e may be calculated from the total pressure equality (7) as

$$\frac{\rho_0}{\rho_e} = \frac{2C_e^2}{\gamma C_A^2 + 2C_s^2}. \tag{28}$$

We suppose here, that the external sound speed $C_e = \sqrt{\gamma p_e/\rho_e}$ is given. In the linear limit $\mu \to 0$ we can integrate the left hand sides of Eqs. (10)-(16) and write all perturbations in terms of ρ and v_2,

$$p = C_s^2 \rho, \quad v_r = v_1 = -\frac{C_s^2}{\rho_0}\int \frac{\partial \rho}{\partial s} dt, \quad B = -\frac{4\pi C_s^2}{B_0}\rho,$$

$$\theta = \int \frac{\partial v_2}{\partial s} dt, \tag{29}$$

with which the terms S_1 to S_7 can be written

$$S_1 = v_2 \frac{\partial v_2}{\partial s} - \frac{C_s^4}{\rho_0^2}\int \frac{\partial \rho}{\partial s} dt \int \frac{\partial^2 \rho}{\partial s^2} dt + \frac{C_s^2}{\rho_0^2}\rho \frac{\partial \rho}{\partial s},$$

$$S_2 = \frac{1}{\rho_0 + \rho_e}$$

$$\left[2C_s^2 \int \frac{\partial \rho}{\partial s} dt \frac{\partial v_2}{\partial s} - (2C_s^2 + C_k^2)\rho \int \frac{\partial^2 v_2}{\partial s^2} dt\right],$$

$$S_3 = \frac{C_s^2}{\rho_0}\frac{\gamma - 1}{2}\frac{\partial \rho^2}{\partial t},$$

$$S_4 = \frac{C_s^4}{C_T^2 \rho_0^2}\left[\frac{\partial}{\partial s}\left(\int \frac{\partial \rho}{\partial s} dt\, \rho\right) - \frac{1}{C_A^2}\frac{\partial \rho^2}{\partial t}\right],$$

$$S_5 = v_2 \int \frac{\partial^2 v_2}{\partial s^2} dt, \quad S_6 = 0, \quad S_7 = -\frac{C_s^4}{2C_A^2 \rho_0}\rho^2. \tag{30}$$

In the linear limit, the right hand sides of the Eqs. (24), (25) can be neglected, and we have two independent wave equations, which describe the propagation of transversal and longitudinal waves in the magnetic flux tube, with velocities C_k and C_T, respectively. Note that the waves are without dispersion in our derivation, because we neglect the perturbations of the external medium. In the considered case (a straight vertical tube without gravitation) the transversal and longitudinal waves propagate independently in the linear limit. The transversal wave corresponds to the propagation of perturbations of the transverse velocity of the fluid in the tube v_2 and of the inclination of the tube θ, while the longitudinal wave is formed by perturbations of density ρ, pressure p, magnetic field B, and the tangential velocities v_1 and v_r. Suppose that the perturbations are proportional to $\exp(i\omega t - iks)$, where ω is the frequency and k the wavenumber. In this case, the frequency and wavenumber are connected by the relation

$$\omega = C_k k, \tag{31}$$

for the transversal, and

$$\omega = C_T k, \tag{32}$$

for the longitudinal wave.

4. Nonlinear interactions of the waves

We now assume that μ is no longer infinitesimally small and consider a mildly nonlinear situation. With μ not too small, the right hand sides of Eqs. (24) and (25) will contribute appreciably.

Consider the longitudinal wave in Eq. (25). Inspection of Eqs. (30) shows that the terms S_3, S_4 and S_7 appearing on the right hand side of Eq. (25) contain perturbations $\sim \rho^2$. This means that the perturbations ρ will nonlinearly and resonantly create second harmonics. Wave energy will be transferred to higher frequencies which leads to a sharpening of wave fronts and thus to shock formation. A given longitudinal wave with the frequency ω and wave number k will thus resonantly excite a longitudinal wave with a doubled frequency and wave number. Due to the absence of dispersion, the new wave will be a mode of the medium, and may be a source for a next generation of waves with still higher frequencies. This type of shock formation has been studied numerically by Herbold et al. (1985).

Another type of longitudinal wave generation is by mode-coupling. The terms S_1 and S_5 on the right hand side of Eq. (25) contain perturbations $\sim v_2^2$. This type of nonlinear nonresonant wave generation, where a pure transversal wave generates longitudinal waves, has been studied numerically by Ulmschneider et al. (1991).

We now consider the transversal wave in Eq. (24). Here inspection of Eqs. (30) shows that the surviving term S_2 contains the perturbation only as product ρv_2. Thus in the absence of a longitudinal perturbation ρ, a direct nonlinear steepening of the transversal wave via the generation of harmonics is not possible. The steepening of the transversal wave is due to the combined nonlinear interaction of longitudinal and transversal waves. This type of transversal (and longitudinal) shock formation has been studied numerically by Zhugzhda et al. (1995).

The nonlinear terms on the right hand sides of Eqs. (24) and (25) show the possibility of three types of nonlinear interaction: the nonresonant formation of longitudinal waves out of transversal waves and two types of resonant wave generation processes: the nonlinear generation of the second harmonic of the longitudinal wave, and the three-wave interaction of two transversal and one longitudinal waves.

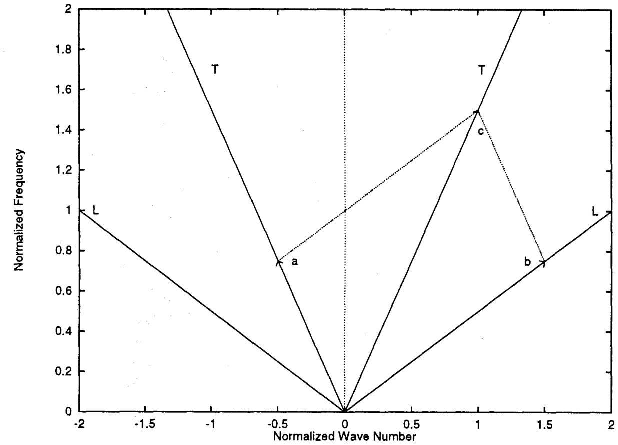

Fig. 1. Frequency (ω)–wavevector (k) diagram for a resonant triplet of interacting waves for the case $C_k > C_T$. L and T indicate loci of longitudinal and transversal modes, respectively. The different interacting waves are labeled a, b and c.

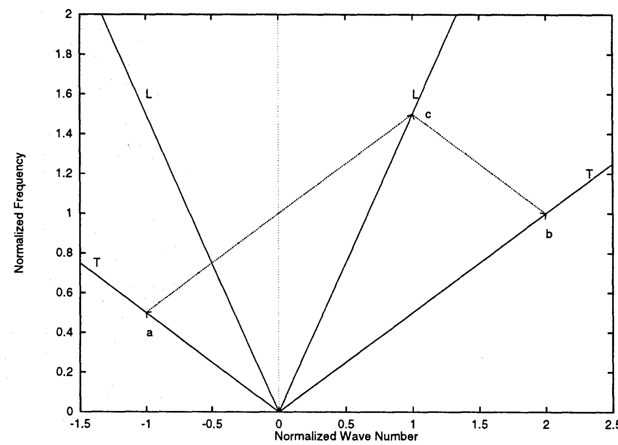

Fig. 2. Same as Fig. 1, however for the case $C_T > C_k$.

Note that in terms of wave interactions, the resonant generation of the second harmonics of the longitudinal wave may be considered as a degenerate resonant three-wave interaction, where two of the three interacting waves are identical. Note that the process is possible only for the longitudinal mode.

Let us now discuss the last process (resonant three-wave interaction of two transversal and one longitudinal wave) in greater detail. The frequencies and wavenumbers of the interacting waves are connected by the resonant conditions:

$$\begin{cases} \omega_a + \omega_b = \omega_c \\ k_a + k_b = k_c \end{cases}, \tag{33}$$

where the indices a, b, c denote different interacting waves; the index c corresponds to the wave with the highest frequency. Consider two different cases: $C_k > C_T$ and $C_k < C_T$.

4.1. Case $C_k > C_T$

For a resonant three-wave interaction which generates waves of higher harmonics, we know from our development above, that a transversal wave results from the interaction of a longitudinal and a transversal wave, while a longitudinal wave from the interaction of two longitudinal waves (excluded here as already discussed above) or two transversal waves. The resonant conditions (33) show that specifying the frequency or wave number of one of the interacting waves determines the frequencies and wave numbers of the other two waves. Fig. 1, for the case $C_k > C_T$, shows the frequency ω-wavenumber k diagram for the wave triplet a, b, c. The loci of the waves with phase speed C_k are labeled T and those with phase speed C_T, are denoted by L. A particular wave is indicated by a vector in the $\omega - k$ diagram on its respective locus. Eq. (33) describes a vector addition. It is seen that in the case of Fig. 1, where $C_k > C_T$ the only possibility is, that the wave c of the highest frequency is a transversal wave which in turn is composed of a longitudinal wave propagating in the same direction and a transversal wave propagating in the opposite direction. Thus, the indices a, c correspond to the transversal waves and index b denotes the longitudinal wave. Using relations (31)-(32), for a given k_c, we derive the wave numbers and frequencies of the other waves of the resonant triplet (33):

$$\omega_c = C_k k_c, \quad \omega_a = -C_k k_a, \quad \omega_b = C_T k_b,$$

$$k_a = \frac{\omega_c}{C_k}\frac{C_T - C_k}{C_T + C_k}, \quad k_b = \frac{2\omega_c}{C_k + C_T}. \tag{34}$$

We now look for the solution of Eqs. (24), (25) in the form

$$\rho = R(\mu s, \mu t)\exp(i\omega_b t - ik_b s) + \text{c.c.},$$

$$v_2 = A_a(\mu s, \mu t)\exp(i\omega_a t - ik_a s)$$

$$+A_c(\mu s, \mu t)\exp(i\omega_c t - ik_c s) + \text{c.c.}, \tag{35}$$

where c.c. means complex conjugate terms. The expansion (35) is well-known as the method of slowly varying amplitudes (Weiland & Wilhelmsson, 1977). Substituting (35) into Eqs. (24), (25), taking into account the resonant conditions (33), and neglecting the terms of third order in μ, we obtain the reduced set of equations for the complex amplitudes of the interactive waves:

$$\frac{\partial A_a}{\partial t} - C_k\frac{\partial A_a}{\partial s} = i\sigma_a A_c R^*,$$

$$\frac{\partial R}{\partial t} + C_T\frac{\partial R}{\partial s} = i\sigma_b A_c A_a^*, \tag{36}$$

$$\frac{\partial A_c}{\partial t} + C_k\frac{\partial A_c}{\partial s} = i\sigma_c A_a R,$$

where the coefficients of the nonlinear interaction are

$$\sigma_a = \frac{1}{2(\rho_0+\rho_e)}\left[2C_s^2\frac{k_b k_c}{\omega_b} - (2C_s^2 + C_k^2)\frac{k_c^2}{\omega_c}\right],$$

$$\sigma_b = \frac{\rho_0 C_A^2}{2\omega_b(C_A^2 + C_s^2)}\left[\omega_b\left(\frac{k_c^2}{\omega_c} - \frac{k_a^2}{\omega_a}\right) + k_b(k_a - k_c)\right], \tag{37}$$

$$\sigma_c = \frac{1}{2(\rho_0+\rho_e)}\left[\frac{2C_s^2 k_b k_a}{\omega_b} - (2C_s^2 + C_k^2)\frac{k_a^2}{\omega_a}\right].$$

The set (36) is well-known in the different branches of physics (see e.g. Weiland & Wilhelmsson 1977). In the case considered the coefficients $\sigma_{a,b,c}$ are all negative and decrease linearly with growing frequency or wave number of any one of the interacting waves. If the frequency or the wave number tends to zero, the coefficients also go to zero.

4.2. Case $C_T > C_k$

Analogous to the reasoning in the previous section, we find, that the wave with the highest frequency is longitudinal, and that this wave interacts with two transversal waves. One of these waves propagates in the positive direction, and other propagates in negative. Thus the indices a, b correspond to the transversal waves and index c corresponds to the longitudinal wave. The expressions for the frequencies and wavenumbers of the interacting waves by means of a given k_c are

$$\omega_c = C_T k_c, \quad \omega_a = -C_k k_a, \quad \omega_b = C_k k_b,$$

$$k_a = \frac{1}{2}\left(\frac{\omega_c}{C_T} - \frac{\omega_c}{C_k}\right), \quad k_b = \frac{1}{2}\left(\frac{\omega_c}{C_T} + \frac{\omega_c}{C_k}\right). \tag{38}$$

The wave with index a propagates opposite to both of the others, as in the previous case. The resonant triplet is shown in Fig. 2. In this case, we look for the solution of Eqs. (24), (25) in the form

$$\rho = R(\mu s, \mu t)\exp(i\omega_c t - ik_c s) + \text{c.c.},$$

$$v_2 = A_a(\mu s, \mu t)\exp(i\omega_a t - ik_a s) +$$

$$A_b(\mu s, \mu t)\exp(i\omega_b t - ik_b s) + \text{c.c.}. \tag{39}$$

Substituting (39) into Eqs. (24), (25), taking into account the resonant conditions (38) and neglecting the terms of third order in μ, we obtain the set of equations for the complex amplitude of the interacting waves:

$$\frac{\partial A_a}{\partial t} - C_k\frac{\partial A_a}{\partial s} = i\sigma_a R A_b^*,$$

$$\frac{\partial A_b}{\partial t} + C_k\frac{\partial A_b}{\partial s} = i\sigma_b R A_a^*, \tag{40}$$

$$\frac{\partial R}{\partial t} + C_T\frac{\partial R}{\partial s} = i\sigma_c A_a A_b,$$

where

$$\sigma_a = \frac{1}{2(\rho_0+\rho_e)}\left[(2C_s^2 + C_k^2)\frac{k_b^2}{\omega_b} - 2C_s^2\frac{k_c k_b}{\omega_c}\right],$$

$$\sigma_b = \frac{1}{2(\rho_0+\rho_e)}\left[(2C_s^2 + C_k^2)\frac{k_a^2}{\omega_a} - 2C_s^2\frac{k_c k_a}{\omega_c}\right], \tag{41}$$

$$\sigma_c = -\frac{\rho_0 C_A^2}{\omega_c(C_A^2 + C_s^2)}\left[k_c(k_a + k_b) - \omega_c\left(\frac{k_a^2}{\omega_a} + \frac{k_b^2}{\omega_b}\right)\right].$$

5. The behaviour of the wave in the tube

It is essential that the coefficients σ_a, σ_b and σ_c are of the same sign for both cases considered. This is no surprise, because all of the waves considered have a positive energy. The sets (36) and (40) have analytical solutions in terms of elliptical functions (see e.g. Weiland & Wilhelmsson 1977). Here we limit ourselves to consider several specific cases. Consider the temporal behaviour ($\partial/\partial x = 0$) of the interacting waves, if the amplitude of one wave is fixed, that is, if we have a wave source with a fixed amplitude.

i) The case $C_k > C_T$: If A_c = const, one gets from (36)

$$\frac{\partial^2 R}{\partial t^2} - \sigma_a\sigma_b|A_c|^2 R = 0,$$
$$\frac{\partial^2 A_a}{\partial t^2} - \sigma_a\sigma_b|A_c|^2 A_a = 0. \quad (42)$$

Thus, in this case, if the amplitude of the transversal wave c is given, then the amplitude of the transversal wave propagating in the opposite direction, and that of the longitudinal wave, grow exponentially. This is a decay instability. The growth rate δ is determined by the nonlinear coefficients and the energy of the wave with the fixed amplitude

$$\delta = \sqrt{\sigma_a\sigma_b|A_c|^2}. \quad (43)$$

If the amplitude of the longitudinal wave is given, R = const, then the temporal dependence of other waves is as follows:

$$\frac{\partial^2 A_a}{\partial t^2} + \sigma_a\sigma_c|R|^2 A_a = 0,$$
$$\frac{\partial^2 A_c}{\partial t^2} + \sigma_a\sigma_c|R|^2 A_c = 0. \quad (44)$$

We see that there are oscillations of the amplitudes of the transversal waves with the period

$$T = \frac{2\pi}{\sqrt{\sigma_a\sigma_c|R|^2}}. \quad (45)$$

Note, that the oscillations are a slow modulation with respect to the wave oscillations with frequencies ω_a and ω_c.

ii) The case $C_k < C_T$: In this case the presence of the source of the longitudinal wave (R = const) leads to an exponential growth of the amplitudes of the transversal waves with the increment

$$\delta = \sqrt{\sigma_a\sigma_b|R|^2}. \quad (46)$$

On the other hand, the fixing of the transversal wave amplitude A_b = const leads to oscillations both of the amplitudes of the transversal wave which moves in the opposite direction, and the longitudinal wave propagating in the same direction with the initial wave. The period of these oscillations is given by

$$T = \frac{2\pi}{\sqrt{\sigma_a\sigma_c|A_b|^2}}. \quad (47)$$

An investigation of the spatial dependence of the interacting wave amplitudes may be carried out in the same manner.

6. Summary and conclusions

Neglecting gravity, we have investigated analytically the nonlinear interaction of longitudinal and transversal waves in thin magnetic flux tubes. Perturbing the magnetohydrodynamic equations in the thin flux tube approximation as given by Cheng (1992), and retaining only terms up to second order, wave equations for the longitudinal and transversal waves were derived which include terms describing the nonlinear interaction of these waves. Both resonant and nonresonant wave generation due to these nonlinear terms was investigated with the aim to better understand our previous numerical work.

1. Writing all perturbations in terms of a longitudinal (density) perturbation ρ and a transversal (velocity) perturbation v_2, we find that the interaction terms for the longitudinal wave contain nonlinear terms $\sim \rho^2, v_2^2$, while the transversal wave has only those $\sim \rho v_2$.

2. The generation of higher harmonics by these nonlinear terms clarifies why in the numerical longitudinal wave computations (Herbold et al. 1985) we have steepening of the longitudinal wave and for purely transverse numerical wave excitations (Ulmschneider et al. 1991) we obtain nonresonant longitudinal wave generation.

3. The direct generation of transversal shocks due to the formation of higher harmonics is impossible if only transversal waves are present. Yet the steepening of the transversal wave is possible if both longitudinal and transversal waves are present, clarifying the numerical transverse shock formation computations by Zhugzhda et al (1995).

4. The resonant nonlinear three-wave interaction of MHD tube waves was investigated in terms of a normal mode analysis and the growth and decay of the wave amplitudes were discussed. It was found that there must always be a longitudinal and two transverse waves present. The frequencies and wave numbers of the interacting waves are connected by the resonant conditions. The change of the wave number or frequency of any one wave leads to the change of the frequencies and wavenumbers of the other waves in the resonant triplet.

5. In the case $C_k > C_T$ the wave with highest frequency is transversal and the other waves consist of a longitudinal wave in the same and a transversal wave in the opposite direction, while in the case $C_T > C_k$ the wave of highest frequency is longitudinal and the others are two oppositely propagating transversal waves.

6. A set of reduced equations, which describes the dynamics of the complex amplitudes of the interacting waves, was obtained. The absolute values of the coefficients of the nonlinear interactions depend upon the frequency or wave number of the incident wave. In the both cases, the absolute values of the coefficients increase with growing frequency of the highest frequency wave. The efficiency of nonlinear interaction, which may be defined as the absolute value of the coefficients of interaction, is stronger in the case $C_k > C_T$, and depends, of course, on the amplitudes of the interacting waves.

7. In realistic situations, where there is a MHD wave noise background, the interactions are possible also in the absence of some

of the waves in the resonant triplet. In this case the required wave number and frequency will be picked out of the noise spectrum and amplified.

8. An analysis of the set of reduced equations for the complex amplitudes of the interacting waves shows either a possibility of a decay instability of the waves if there is a source for a highest frequency wave, or a formation of long period pulsations (with respect to the periods of the interacting waves) if there is a source for one of the two low frequency waves. The increment of the decay instability and the period of the pulsations are determined, in particular, by the initial amplitudes of interacting waves.

9. For the formation of the transversal shock wave by Zhugzhda et al. (1995) we suggest that the wave energy is transferred to the high frequency part of the spectrum due to a cascade of the three-wave interaction, which includes the longitudinal wave. The initial transversal wave (combined with the other transversal wave of possibly small amplitude) excites the longitudinal wave, while the longitudinal wave excites the transversal wave with higher frequency, etc.

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