Assignment #3: due Tuesday, Nov. 6

Theoretical Astrophysics

Winter 2006/2007
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1. **Degenerate electron gas** 30 pt

Assume that for a degenerate electron gas all quantum levels up to the Fermi momentum p_F are filled and all others are empty, so that the phase-space probability density is given by

$$f(\vec{q},\vec{p}) = \begin{cases} 2/h^3 & \text{for} \quad p \le p_F \\ 0 & \text{for} \quad p > p_F \,. \end{cases} \tag{1}$$

The factor 2 takes into account the two spin orientations of the electrons.

Use this distribution function to compute the density and pressure and show that this gas has a polytropic equation of state

$$P = K\,\rho^\gamma \,. \tag{2}$$

Find the index γ and the constant K, assuming each electron is accompanied by one proton whose kinetic energy is negligible.

2. **Elementary hydro and thermodynamics** 20 pt

In the absence of viscosity and external forces, the continuity, Euler and energy equations for an ideal gas can be written

$$\frac{\partial \rho}{\partial t} + \nabla \cdot (\rho\,\vec{v}) = 0 \tag{3}$$

$$\frac{\partial \vec{v}}{\partial t} + (\vec{v}\cdot\nabla)\,\vec{v} = -\frac{1}{\rho}\,\nabla P \tag{4}$$

$$\rho\left(\frac{\partial \varepsilon}{\partial t} + (\vec{v}\cdot\nabla)\,\varepsilon\right) + P\,\nabla\cdot\vec{v} = 0\,, \tag{5}$$

where ρ, $\vec{v}$, P, and ε are the density, velocity, pressure and the internal energy per unit mass, respectively.

(a) By rewriting equations (4) and (5) in conservative form, i.e.,

$$\frac{\partial}{\partial t}(\text{conserved quantity}) + \nabla\cdot(\text{corresponding flux}) = 0 \tag{6}$$

find expression for the fluxes of momentum and energy. *Hint: Start with the Eulerian time derivative of the conserved quantities, i.e. momentum density and total kinetic energy density, and find the conservation law by successively using the hydrodynamic equations as given above. Note also that using the index notation will be useful when performing the differentiations.*

(b) For an ideal gas ($\varepsilon = c_V\, T$, $P = \rho T\, k/m$) show that

$$h = \frac{\gamma}{\gamma - 1} \frac{P}{\rho} \tag{7}$$

where γ is the ratio of the specific heat at constant pressure c_P and constant volume c_V.

3. **Virial Theorem** 5 pt

Consider an isolated, non-rotating, self-gravitating system containing an ideal gas in virial equilibrium in its center-of-mass reference frame. The kinetic energy integral in the scalar virial equation is

$$U = (\gamma - 1)\, \frac{3}{2}\, U_{\text{int}}, \tag{8}$$

where U_{int} is the internal energy of the gas and γ its adiabatic index. Which values of γ permit bound configurations (i.e., configurations with negative total energy)?