Assignment #7: due Tuesday, Dec. 4

Theoretical Astrophysics

Winter 2006/2007
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1. **Parker instability** 30 pt

Consider an isothermal gas in the galactic disk which is threaded with a horizontal magnetic field. Assume a constant gravitational field perpendicular to the disk plane in the z direction, i.e. $\vec{g} = -\hat{z}\, g$ and a magnetic field parallel to the disk plane x which varies only with z, i.e. $B = \hat{x}\, B(z)$. For simplicity study the system in two dimensions using cartesian coordinates.

(a) Assume that the system is in hydrostatic equilibrium with a constant ratio of the magnetic to thermal pressure, i.e.

$$\alpha \equiv \frac{B^2}{8\pi\, P} = \text{const.} \tag{1}$$

What is the pressure distribution as a function of z? Use the relation $P = c_\mathrm{s}^2\, \rho$ where c is the constant speed of sound and the scale height $H = (1+\alpha)\, c^2/g$ to express the result.

Now consider this system slightly perturbed out of its equilibrium. Then, from the linear perturbation analysis one gets the following dispersion relation in the xz-plane,

$$n^4 \;+\; c_\mathrm{s}^2 \left[(1+2\alpha)\left(k^2 + \frac{k_0^2}{4}\right)\right] n^2 + k_x^2\, c_\mathrm{s}^4 \left[2\alpha\, k^2 + k_0^2 \left[\left(1 + \frac{3\alpha}{2}\right) - (1+\alpha)^2\right]\right] = 0\;, \tag{2}$$

(you might want to derive this relation in your spare time), where $n = i\omega$, $k_0 = H^{-1}$, and the Fourier modes in the x and z direction for the perturbed quantities are

$$\exp\left(i\omega\, t + ik_x\, x\right) \quad , \quad \exp\left(i\omega\, t + ik_z\, z\right) \tag{3}$$

with $k^2 = k_x^2 + k_z^2$.

(b) Show that in the absence of a magnetic field all roots (in terms of n^2) of this dispersion relation are negative, i.e. $n^2 < 0$. What is the physical implication of this result regarding the instability?

(c) In the case of a non-vanishing magnetic field derive the instability criterion for the Parker instability (magnetic Rayleigh-Taylor instability)

$$\left(\frac{k}{k_0/2}\right)^2 < 2\alpha + 1 \, . \tag{4}$$

Hint: Use the roots of n^2 to find at leat one unstable mode, i.e. $n^2 < 0$.

(d) Show that the instability criterion is equivalent to

$$\lambda_x \;>\; \Lambda_x \equiv 4\pi\, H \left[\frac{1}{2\alpha+1}\right]^{1/2} \tag{5}$$

$$\lambda_z \;>\; \Lambda_z \equiv \frac{\Lambda_x}{\left(1-(\Lambda_x/\lambda_x)^2\right)^{1/2}} \, , \tag{6}$$

with the wavelengths $\lambda_x = 2\pi/k_x$, and $\lambda_z = 2\pi/k_z$.