Assignment #8: due Tuesday, Dec. 11

Theoretical Astrophysics

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1. **Magnetic Braking of an Alligned Rotor** 40 pt

Consider a uniform gaseous disk of density $\rho_{\rm cl}$ and half-thickness Z rotating rigidly with an initial angular velocity Ω_0. Furthermore assume the disk is threaded with a magnetic field $\vec{B}$ of strength B_0, initially uniform and parallel to the rotation axis. The magnetic field links the disk with the external medium of density $\rho_{\rm ext}$ which is initially at rest. Assume axisymmetry and use cylindrical coordinates (R, φ, z) for the calculations.

(a) Derive the evolution equations for the toroidal magnetic field B_φ and the angular velocity Ω (where $v_\varphi = R\,\Omega$) outside of the disk, i.e. $|z| > Z$, assuming that the radial velocity v_r and the poloidal velocity v_z are negligible small (compared to the Alfvén velocity).

(b) Show that the evolution of the external medium can be expressed by the wave equation

$$\frac{\partial^2 \Omega}{\partial t^2} = v_{\rm A,ext}^2 \frac{\partial^2 \Omega}{\partial z^2} \tag{1}$$

where $v_{\rm A,ext} = B_0/\sqrt{4\pi\,\rho_{\rm ext}}$ is the Alfvén velocity in this medium.

(c) Derive the evolution equation of the angular velocity at the surface of the disk, i.e. $|z| = Z$, using the torque per unit area, $N = R\,B_0\,B_\varphi/4\pi$, which the magnetic field exerts on the surface of the disk. Result:

$$\frac{\partial^2 \Omega_{\rm cl}}{\partial t^2} = \frac{1}{Z}\frac{\rho_{\rm ext}}{\rho_{\rm cl}} v_{\rm A,ext}^2 \left.\frac{\partial \Omega}{\partial z}\right|_{|z|=Z} \tag{2}$$

(d) Combine equations (1) and (2) to calculate the spin-down time of the disk. [Use the solution of equation (1) at the disk surface $|z| = Z$.]

2. **A Simple Approach to Magnetorotational Instability** 35 pt

The simplest system that displays the magnetorotational instability is an axis-symmetric differentially rotating gaseous disk in the presence of a weak vertical magnetic field, i.e. $\vec{B}_0 = B_z \vec{e}_z$. We assume the disk is initially homogeneous with density ρ_0 and has no radial or vertical motions, i.e. $\vec{v}_0 = v_\varphi \vec{e}_\varphi$. In the following we consider a fluid element that is displaced from its appropriate circular orbit by a small distance $\vec{x} = x_R \vec{e}_R + x_\varphi \vec{e}_\varphi$, where the displacement follows a vertical oscillation $\propto \exp(ikz)$. We neglect the effects of viscosity and study time evolution of the system with perturbations induced as described above. As in the previous assignment we use a cylindrical coordinate system.

(a) Derive the equations of magneto-hydrodynamics for this system to linear order for the quantities $\delta\rho$, δP, $\delta\vec{u}$, and $\delta\vec{B}$, which are perturbed density, pressure, velocity and magnetic field, respectively. Consider the velocity $\vec{u}$ which denotes the deviation of the true fluid velocity $\vec{v}$ at any location from the azimuthal circular velocity $R\Omega(R)\vec{e}_\varphi$,

$$u_R = v_R \ , \qquad u_\varphi = v_\varphi - R\Omega(R) \ , \qquad u_z = v_z \ , \tag{3}$$

and where $\Omega(R)$ is the circular velocity at radius R.
The result is,

$$-\omega\frac{\delta\rho}{\rho} + k\delta u_z = 0 \ , \tag{4}$$
$$-i\omega\delta u_R - 2\Omega\delta u_\varphi - i\frac{kB_z}{4\pi\rho}\delta B_R = 0 \ , \tag{5}$$
$$-i\omega\delta u_\varphi + \frac{\kappa^2}{2\Omega}\delta u_R - i\frac{kB_z}{4\pi\rho}\delta B_\varphi = 0 \ , \tag{6}$$
$$-\omega\delta u_z + k\frac{\delta P}{\rho} = 0 \ , \tag{7}$$
$$-\omega\delta B_R = kB_z\delta u_R \ , \tag{8}$$
$$-i\omega\delta B_\varphi = ikB_z\delta u_\varphi \ , \tag{9}$$
$$\delta B_z = 0 \ , \tag{10}$$

where κ is the epicyclic frequency,

$$\kappa^2 = \frac{1}{R^3}\frac{d(R^4\Omega^2)}{dR} \ , \tag{11}$$

which usually has values in the range $\Omega \leq \kappa \leq 2\Omega$. To close this set of equations we take the usual equation of state,

$$\frac{\delta P}{P} = \gamma\frac{\delta\rho}{\rho} \ . \tag{12}$$

(b) Using equations (4) to (12) relate $\delta\vec{B}$ to the displacement vector $\vec{x}$. (Recall that $d\vec{x}/dt = \vec{u}$ and $d/dt \to i\omega$ in Fourier space.)

(c) We now can substitute the magnetic field terms in the equations of motion. Show that these equations describe the R and φ component of the displacement vector as a set of coupled damped oscillators. Use these equations to find the criterion for instability. Discuss your result.