Assignment #9: due Tuesday, Dec. 18

Theoretical Astrophysics

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In a cold, magnetised plasma consisting of electrons (charge $q_{\rm e} = -e$, mass $m_{\rm e}$) and ions (charge $q_{\rm i} = Z\,e$, mass $m_{\rm i}$), the equation govering the propagation of a wave-like disturbance, $\vec{E} = \vec{E}_0 \exp\left(i\vec{k}\cdot\vec{x} - i\omega t\right)$ is

$$\Lambda\,\vec{E} = 0\,. \tag{1}$$

We use cartesian coordinates with basis $(\vec{e}_x, \vec{e}_y, \vec{e}_z)$ and assume that the wave propagates along the magnetic field which we take parallel to $\vec{e}_z$, the matrix Λ is

$$\Lambda = \begin{pmatrix} S - n^2 & -iD & 0 \\ iD & S - n^2 & 0 \\ 0 & 0 & P \end{pmatrix} \tag{2}$$

where $n = kc/\omega$ is the refractive index, and

$$S = 1 - \frac{\omega_{\rm pe}^2}{\omega^2 - \Omega_{\rm e}^2} - \frac{\omega_{\rm pi}^2}{\omega^2 - \Omega_{\rm i}^2}\,, \tag{3}$$

$$D = \frac{\omega_{\rm pe}^2\,\Omega_{\rm e}}{\omega(\omega^2 - \Omega_{\rm e}^2)} + \frac{\omega_{\rm pi}^2\,\Omega_{\rm i}}{\omega(\omega^2 - \Omega_{\rm i}^2)}\,, \tag{4}$$

$$P = 1 - \frac{\omega_{\rm pe}^2}{\omega^2} - \frac{\omega_{\rm pi}^2}{\omega^2}\,. \tag{5}$$

The quantities $\omega_{\rm pe,pi} = \sqrt{4\pi\, n_{\rm e,i}\, q_{\rm e,i}^2/m_{\rm e,i}}$ are the electron and ion plasma frequencies (with $n_{\rm e,i}$ the number densities) and $\Omega_{\rm e,i} = q_{\rm e,i}B/m_{\rm e,i}\,c$ are the electron and ion gyration frequencies. Note that both have opposite signs.

1. **Alfvén waves** 30 pt

   (a) Find the dispersion relation for the transversal waves in a neutral electron-proton plasma in the frequency range $\omega \ll \Omega_{\rm i}$ and $\omega \ll \omega_{\rm pi}$. Make use of $m_{\rm i} \gg m_{\rm e}$.

   (b) Find the polarisation vectors of the transversal modes in this frequency range and classify them as linear, circular or elliptical.

   (c) Calculate the group and phase velocities of these waves.

2. **Faraday rotation** 30 pt

(a) For $\omega \gg \omega_{\mathrm{pe,pi}}$ and $\omega \gg \Omega_{\mathrm{e,i}}$, show that the dispersion relation in an electron-proton plasma for waves travelling in the positive z direction can be written approximately as

$$\frac{k\,c}{\omega} = 1 - \frac{\omega_{\mathrm{pe}}^2 + \omega_{\mathrm{pi}}^2}{2\omega^2} \pm \frac{\omega_{\mathrm{pe}}^2\,\Omega_{\mathrm{e}}}{2\omega^3} \tag{6}$$

(use the fact that $m_{\mathrm{i}} \gg m_{\mathrm{e}}$) where the upper and lower signs refer to the polarisation vectors $(1/\sqrt{2}, \pm i/\sqrt{2}, 0)$.

(b) Show that a linearly polarised photon that is emitted along the magnetic field will rotate its direction of polarisation as it propagates by an amount proportional to the inverse square of its frequency.

(c) For ionised hydrogen gas in the Galactic plane with $n = 1\,\mathrm{cm}^{-3}$ and $B = 20\,\mu G$, find the distance over which a photon of frequency 3 GHz that is emitted linearly polarised in the x direction must travel before converting completely to one polarised in the y direction. Assume propagation along a uniform magnetic field.

(d) How does the result change if the photon propagates in a hypothetical electron-positron plasma ($m_{\mathrm{e}} = m_{\mathrm{i}}$)?