Assignment #3: due Thursday, Oct. 30

Theoretical Astrophysics

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1. **Elementary hydro and thermodynamics** 20 pt

In the absence of viscosity and external forces, the continuity, Euler and energy equations for an ideal gas can be written

$$\frac{\partial \rho}{\partial t} + \nabla \cdot (\rho \vec{v}) = 0 \tag{1}$$

$$\frac{\partial \vec{v}}{\partial t} + (\vec{v} \cdot \nabla) \vec{v} = -\frac{1}{\rho} \nabla P \tag{2}$$

$$\rho \left(\frac{\partial \varepsilon}{\partial t} + (\vec{v} \cdot \nabla) \varepsilon \right) + P \nabla \cdot \vec{v} = 0 \,, \tag{3}$$

where ρ, $\vec{v}$, P, and ε are the density, velocity, pressure and the internal energy per unit mass, respectively.

(a) By rewriting equations (2) and (3) in conservative form, i.e.,

$$\frac{\partial}{\partial t}(\text{conserved quantity}) + \nabla \cdot (\text{corresponding flux}) = 0 \tag{4}$$

find expression for the fluxes of momentum and energy. *Hint: Start with the Eulerian time derivative of the conserved quantities, i.e. momentum density and total kinetic energy density, and find the conservation law by successively using the hydrodynamic equations as given above. Note also that using the index notation will be useful when performing the differentiations.*

(b) For an ideal gas ($\varepsilon = c_V T$, $P = \rho T k/m$) show that

$$h = \frac{\gamma}{\gamma - 1} \frac{P}{\rho} \tag{5}$$

where γ is the ratio of the specific heat at constant pressure c_P and constant volume c_V.

2. **Virial Theorem** 5 pt

Consider an isolated, non-rotating, self-gravitating system containing an ideal gas in virial equilibrium in its center-of-mass reference frame. The kinetic energy integral in the scalar virial equation is

$$U = (\gamma - 1)\,\frac{3}{2}\,U_{\rm int}, \tag{6}$$

where $U_{\rm int}$ is the internal energy of the gas and γ its adiabatic index. Which values of γ permit bound configurations (i.e., configurations with negative total energy)?

3. **Virial Theorem with surface terms** 20 pt

The scalar virial theorem with surface terms included can be written as

$$\frac{1}{2}\ddot{I} = 2(T - T_s) + 2(U - U_s) + W - \frac{1}{2}\dot{\Psi}_s \tag{7}$$

where

$$T_s \equiv \frac{1}{2}\int_S \vec{r}\cdot\rho\vec{v}\vec{v}\cdot d\vec{S}, \tag{8}$$

$$U_s \equiv \frac{1}{2}\int_S P\vec{r}\cdot d\vec{S}, \tag{9}$$

and

$$\dot{\Psi}_s \equiv \frac{d}{dt}\int_S (\rho\vec{v}r^2)\cdot d\vec{S}. \tag{10}$$

(a) Give a physical explanation for the origin of the T_s, U_s and $\dot{\Psi}_s$ terms.

(b) Consider a homogeneous, spherical cloud of mass M and radius R. Show that if the time dependent terms in Eq. 7 are zero, and if $T \gg T_s$, then we can construct the following expression for the thermal pressure acting on the surface of the cloud:

$$P_s = \frac{1}{4\pi}\left(-\frac{3}{5}\frac{GM^2}{R^4} + 3\frac{c_s^2 M}{R^3} + \frac{\sigma^2 M}{R^3}\right), \tag{11}$$

where c_s is the isothermal sound speed, and σ is the 1D internal velocity dispersion. [Note: $T = (1/2)M\sigma^2$].

(c) We can define a "gravitational pressure" P_G as

$$P_G \equiv -\frac{W}{3V} \tag{12}$$

where V is the volume of the cloud. Show, using Eq. 11, that the total energy of the cloud can be written as

$$E = \frac{3}{2}\left(P_s - P_G\right)V \tag{13}$$

(d) A giant molecular cloud with mass $M = 10^5\,M_\odot$, radius $R = 10\,\mathrm{pc}$, sound speed $c_s = 0.2\ \mathrm{km\ s^{-1}}$ and velocity dispersion $\sigma = 3.0\ \mathrm{km\ s^{-1}}$ is surrounded by cold atomic gas with thermal pressure $P = 2\times10^4\ \mathrm{K\,cm^{-3}}$. What is the total energy of the cloud?