Assignment #4: due Thursday, Nov. 6

Theoretical Astrophysics

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1. **Parker wind solution** 40 pt

   Consider a steady, radial, adiabatic flow of an ideal gas in the gravitational field of a star. The polytropic law is

   $$P = K\,\rho^{\gamma}\,, \tag{1}$$

   where K is constant along the streamlines and $\gamma < 5/3$.

   (a) Show that the continuity equation can be written as

   $$4\pi\,r^2\,\rho\,v = \dot{M} \tag{2}$$

   where v is the radial velocity and $\dot{M}$ is the constant rate of change of mass. Derive the relevant Euler equation for this spherically symmetric system.

   (b) Show that a smooth solution containing both sub- and supersonic regions of the flow exists only if

   $$v^2 = c_s^2 \quad \text{at} \quad r = \frac{G\,M}{2c_s^2} \tag{3}$$

   where $c_s = \sqrt{\gamma P/\rho}$ is the speed of sound (which depends on r).

   (c) Imposing this condition, and the boundary conditions

   $$\rho = \rho_* \quad \text{and} \quad c_s = c_* \tag{4}$$

   at the surface $r = r_*$ of the star, find the mass loss rate in the wind in the limit of low surface velocity $v_* \ll c_*$, given that the surface temperature is large compared to the "virial" temperature, i.e. $c_*^2 \gg G\,M/r_*$.

   (d) Find the location of the sonic point and discuss the behaviour of the solutions as $\gamma \to 5/3$.

2. **Lane-Emden equation** 30 pt

   In a spherically symmetric system, the equations of hydrostatic equilibrium and Poisson's equation are:

   $$\frac{1}{\rho}\frac{\mathrm{d}P}{\mathrm{d}r} = -\frac{\mathrm{d}\Phi}{\mathrm{d}r} \tag{5}$$

   $$\frac{1}{r^2}\frac{\mathrm{d}}{\mathrm{d}r}\left(r^2\frac{\mathrm{d}\Phi}{\mathrm{d}r}\right) = 4\pi\,G\,\rho \tag{6}$$

   where Φ is the gravitational potential and G Newton's gravitational constant.

(a) Taking $\Phi(r_{\rm surf}) = 0$ and $\rho(r_{\rm surf}) = 0$ at the surface of the star, $r = r_{\rm surf}$, show that for a polytropic equation of state, i.e. $P = K\rho^{(n+1)/n} = K\rho^{\gamma}$, the density in the star ($\Phi < 0$) can be expressed as

$$\rho = \left(\frac{-\Phi}{(n+1)K}\right)^{n} \tag{7}$$

(b) Substitute this expression into the Poisson equation and show that this then reduces to the *Lane-Emden equation*:

$$\frac{1}{z^2}\frac{\mathrm{d}}{\mathrm{d}z}\left(z^2\frac{\mathrm{d}w}{\mathrm{d}z}\right) + w^n = 0 \tag{8}$$

when written in terms of the variables $w = \Phi/\Phi_c$ and $z = r/r_0$, where Φ_c is the potential at $r = 0$ and r_0 is a characteristic length scale. Find an expression for r_0 in terms of n and K.

(c) Given that there exists a solution of the Lane-Emden equation for the given system, show that the radius R of a non-relativistic degenerate star ($n = 3/2$) is related to its total mass by

$$R \propto M^{-1/3}\,. \tag{9}$$