Assignment #6: due Thursday, Nov. 20

Theoretical Astrophysics

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1. **Accretion Disk** 60 pt

Consider a thin (non-selfgravitating), axisymmetric disk in cylindrical coordinates, (R, θ, z), whose evolution can be described by the continuity equation and the Navier-Stokes equation

$$\frac{\partial \vec{v}}{\partial t} + (\vec{v} \cdot \nabla)\, \vec{v} = \vec{g} - \frac{1}{\rho} \nabla P + \vec{\eta} \tag{1}$$

(here given in Cartesian coordinates), where $\vec{\eta}$ is the viscous friction and $\vec{g}$ is the gravitational acceleration due to the central star with mass M. Assume that the velocity does not vary with disk height z and $v_z = 0$.

(a) Use the surface density $\Sigma = \int \rho \, \mathrm{d}z$ to rewrite the continuity equation in the form

$$\frac{\partial \Sigma}{\partial t} + \frac{1}{R} \frac{\partial}{\partial R} (R\, \Sigma\, v_R) = 0 \tag{2}$$

and derive the equation which determines the evolution of the angular momentum

$$\frac{\partial}{\partial t} \Big(\Sigma\, R^2\, \Omega \Big) + \frac{1}{R} \frac{\partial}{\partial r} \Big(\Sigma\, R^3\, \Omega\, v_R \Big) = T \tag{3}$$

where $\Omega = v_\theta / R$ is the angular velocity and the function T involves the viscous terms.

(b) Assume that the viscous force per unit area $f_{\rm visc}$ is proportional to the rotational shear $A = R\, \mathrm{d}\Omega/\mathrm{d}R$, i.e.

$$f_{\rm visc} = \nu\, \rho\, A \tag{4}$$

where ν is the kinetic viscosity, derive the viscous term T.

(c) Now assume that the disk matter is in Keplerian orbits around the central star, i.e. $\Omega = \sqrt{G\, M / R^3}$. Show that the disk evolution is then governed by the equation

$$\frac{\partial \Sigma}{\partial t} = \frac{3}{R} \frac{\partial}{\partial R} \left(R^{1/2} \frac{\partial}{\partial R} (\nu\, \Sigma\, R^{1/2}) \right) . \tag{5}$$

(d) Consider a steady state, thin accretion disk, i.e. $|\nabla P/\rho| \ll G\,M/r^2$, $\partial\vec{v}/\partial t = 0$. Show that the mass infall rate $\dot{m} = -2\pi\,R\,\Sigma\,v_R$ is given by

$$\dot{m} = 3\pi\,\nu\,\Sigma\left[1 - \left(\frac{R_*}{R}\right)^{1/2}\right]^{-1} \tag{6}$$

where R_* are the radii with vanishing shear (say the stellar surface or the inner disk radius).

(e) Given that the energy dissipation per unit area of the disk surface is given by

$$\dot{E} = -\nu\,\Sigma\,R^2\left(\frac{\mathrm{d}\Omega}{\mathrm{d}R}\right)^2, \tag{7}$$

show that the total luminosity emitted from the accretion disk is then

$$L = \frac{G\,M\,\dot{m}}{2\,R_*}. \tag{8}$$

Compare this to the total potential energy released and discuss possible discrepancies.