Assignment #8: due Thursday, December 4, 2008

Theoretical Astrophysics

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1. **Magnetic Braking of an Alligned Rotor** 40 pt

Consider a uniform gaseous disk of density $\rho_{\rm cl}$ and half-thickness Z rotating rigidly with an initial angular velocity Ω_0. Furthermore assume the disk is threaded with a magnetic field $\vec{B}$ of strength B_0, initially uniform and parallel to the rotation axis. The magnetic field links the disk with the external medium of density $\rho_{\rm ext}$ which is initially at rest. Assume axisymmetry and use cylindrical coordinates (R, φ, z) for the calculations.

(a) Derive the evolution equations for the toroidal magnetic field B_φ and the angular velocity Ω (where $v_\varphi = R\,\Omega$) outside of the disk, i.e. $|z| > Z$, assuming that the radial velocity v_r and the poloidal velocity v_z are negligible small (compared to the Alfvén velocity).

(b) Show that the evolution of the external medium can be expressed by the wave equation

$$\frac{\partial^2 \Omega}{\partial t^2} = v_{\rm A,ext}^2 \, \frac{\partial^2 \Omega}{\partial z^2} \tag{1}$$

where $v_{\rm A,ext} = B_0/\sqrt{4\pi\,\rho_{\rm ext}}$ is the Alfvén velocity in this medium.

(c) Derive the evolution equation of the angular velocity at the surface of the disk, i.e. $|z| = Z$, using the torque per unit area, $N = R\,B_0\,B_\varphi/4\pi$, which the magnetic field exerts on the surface of the disk. Result:

$$\frac{\partial^2 \Omega_{\rm cl}}{\partial t^2} = \frac{1}{Z}\,\frac{\rho_{\rm ext}}{\rho_{\rm cl}}\,v_{\rm A,ext}^2 \left.\frac{\partial \Omega}{\partial z}\right|_{|z|=Z} \tag{2}$$

(d) Combine equations (1) and (2) to calculate the spin-down time of the disk. *Hint: Use the solution of equation (1) at the disk surface $|z| = Z$.*

2. **Plasma Waves: Fluid Treatment** 25 pt

A common approximation in plasma physics is that the electrons and/or ions can be described by fluid equations. Assume the ions in a plasma can be treated as a smooth, uniform, motionless background charge density that neutralizes the average electron charge density. The electron number density n and velocity $\vec{v}$ obey the continuity equation and the equation of motion, in which the Lorentz force appears in the same way as an external force,

$$\frac{\partial n}{\partial t} + \vec{\nabla} \cdot (n\vec{v}) = 0\,, \tag{3}$$

$$\frac{\partial \vec{v}}{\partial t} + (\vec{v} \cdot \vec{\nabla})\vec{v} = \frac{1}{n m_\mathrm{e}} \left(-\vec{\nabla} P + qn\vec{E} + \frac{1}{c}\vec{j} \times \vec{B} \right)\,, \tag{4}$$

where m_e is the electron mass, $q = -e$ is the electron charge, P the pressure of the electron gas, $\vec{j}$ the current density, and $\vec{E}$ and $\vec{B}$ are the electric and magnetic field strengths. The electron density and pressure are related by the equation of state,

$$P = Kn^\gamma, \tag{5}$$

where K is a constant and where we assume the electrons behave as an adiabatic gas with $\gamma = 5/3$. Electric and magnetic fields, $\vec{E}$ and $\vec{B}$, are related to the charge density qn and the current $\vec{j}$ via Maxwell's equations.

(a) Assume the system originally is in equilibrium and apply a small perturbation of the form,

$$n = n + 0 + \delta n\,, \qquad P = P_0 + \delta P\,, \qquad \vec{v} = \delta\vec{v}\,,$$

$$\vec{E} = \delta\vec{E}\,, \qquad \vec{B} = \delta\vec{B}\,, \qquad \vec{j} = \delta\vec{j}\,,$$

where n_0 and P_0 are the homogeneous equilibrium density and pressure. Linearize the equations and find the equation that governs the evolution of the density perturbation δn.

(b) Consider plane wave perturbations of the form $\delta n \propto \exp(i\vec{k}\vec{x} - i\omega t)$, derive their dispersion relation, and find their phase and group velocities as function of the thermal velocity of the electrons, $v_\mathrm{th} = (P_0/m_\mathrm{e} n_0)^{1/2}$, and the ratio k/k_D, where

$$k_\mathrm{D} = \left(\frac{4\pi q^2 n_0}{m_\mathrm{e} v_\mathrm{th}^2} \right)^{1/2}$$

is the Debye wave number as introduced in the lecture.

(c) Discuss the nature of this waves in the limits of small and larger wavelengths.