

Assignment #9: due Thursday, Dec. 11

Theoretical Astrophysics

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In a cold, magnetised plasma consisting of electrons (charge $q_e = -e$, mass m_e) and ions (charge $q_i = Ze$, mass m_i), the equation governing the propagation of a wave-like disturbance, $\vec{E} = \vec{E}_0 \exp(i\vec{k} \cdot \vec{x} - i\omega t)$ is

$$\Lambda \vec{E} = 0. \quad (1)$$

We use cartesian coordinates with basis $(\vec{e}_x, \vec{e}_y, \vec{e}_z)$ and assume that the wave propagates along the magnetic field which we take parallel to \vec{e}_z . In this case, the matrix Λ is

$$\Lambda = \begin{pmatrix} S - n^2 & -iD & 0 \\ iD & S - n^2 & 0 \\ 0 & 0 & P \end{pmatrix} \quad (2)$$

where $n = kc/\omega$ is the refractive index, and

$$S = 1 - \frac{\omega_{pe}^2}{\omega^2 - \Omega_e^2} - \frac{\omega_{pi}^2}{\omega^2 - \Omega_i^2}, \quad (3)$$

$$D = \frac{\omega_{pe}^2 \Omega_e}{\omega(\omega^2 - \Omega_e^2)} + \frac{\omega_{pi}^2 \Omega_i}{\omega(\omega^2 - \Omega_i^2)}, \quad (4)$$

$$P = 1 - \frac{\omega_{pe}^2}{\omega^2} - \frac{\omega_{pi}^2}{\omega^2}. \quad (5)$$

The quantities $\omega_{pe,pi} = \sqrt{4\pi n_{e,i} q_{e,i}^2 / m_{e,i}}$ are the electron and ion plasma frequencies (with $n_{e,i}$ the number densities) and $\Omega_{e,i} = q_{e,i} B / m_{e,i} c$ are the electron and ion gyration frequencies. Note that both have opposite signs.

1. Alfvén waves

30 pt

- Find the dispersion relation for the transversal waves in a neutral electron-proton plasma in the frequency range $\omega \ll \Omega_i$ and $\omega \ll \omega_{pi}$. Make use of the fact that $m_i \gg m_e$.
- Find the polarisation vectors of the transversal modes in this frequency range and classify them as linear, circular or elliptical.
- Calculate the group and phase velocities of these waves.

2. Faraday rotation

30 pt

- (a) For $\omega \gg \omega_{pe,pi}$ and $\omega \gg \Omega_{e,i}$, show that the dispersion relation in an electron-proton plasma for waves travelling in the positive z direction can be written approximately as

$$\frac{kc}{\omega} = 1 - \frac{\omega_{pe}^2 + \omega_{pi}^2}{2\omega^2} \pm \frac{\omega_{pe}^2 \Omega_e}{2\omega^3}, \quad (6)$$

(use the fact that $m_i \gg m_e$) where the upper and lower signs refer to the polarisation vectors $(1/\sqrt{2}, \pm i/\sqrt{2}, 0)$.

- (b) Show that a linearly polarised photon that is emitted along the magnetic field will rotate its direction of polarisation as it propagates by an amount proportional to the inverse square of its frequency.
- (c) For ionised hydrogen gas in the Galactic plane with $n = 1 \text{ cm}^{-3}$ and $B = 20 \mu\text{G}$, find the distance over which a photon of frequency 3 GHz that is emitted linearly polarised in the x direction must travel before converting completely to one polarised in the y direction. Assume propagation along a uniform magnetic field.
- (d) How does the result change if the photon propagates in a hypothetical electron-positron plasma ($m_e = m_i$)?