## Assignment #3: due Tuesday, Nov. 10, 2009

# Theoretical Astrophysics

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1. **Scalar Virial Theorem** 15 pt

   (a) Globular clusters are gravitationally bound, dense stellar systems with half-light radii of typically $r_{1/2} \sim 5\,\mathrm{pc}$ and (one-dimensional) velocity dispersions $\sigma_{1/2}$ of roughly $6\,\mathrm{km\,s^{-1}}$. The total bolometric luminosity is $\sim 2.4 \times 10^{38}\,\mathrm{erg\,s^{-1}}$. Using the scalar virial theorem calculate the mass-to-light ratio $\Upsilon$ for a globular cluster in units of the solar mass-to-light ratio $\Upsilon_\odot = M_\odot / L_\odot$.

   (b) The dwarf-spheroidal galaxy Draco - a satellite galaxy of our Milky Way - has a distance of about $d \approx 72\,\mathrm{kpc}$ and an estimated half-mass radius $r_{1/2}$ of about $5.7'$ on the sky. Draco has a luminosity of $L = 2.6 \times 10^5\,L_\odot$ and a (one-dimensional) dispersion velocity of $\sigma \approx 13.2\,\mathrm{km\,s^{-1}}$. Calculate the mass-to-light ratio $\Upsilon$ of the Draco galaxy.

   (c) Compare the result to the one found by Irwin & Hatzidimitriov (1995, Monthly Notices of The Royal Astronomical Society, 277, 1354). Speculate about the difference? The high mass-to-light ratio suggests that Draco is dark matter dominated. Can you think of an alternative explanation for the high value of $\Upsilon$ derived above?

2. **Virial Theorem with surface terms** 20 pt

   The scalar virial theorem with surface terms included can be written as

$$\frac{1}{2}\ddot{I} = 2(T - T_s) + 2(U - U_s) + W - \frac{1}{2}\dot{\Psi}_s \tag{1}$$

   where

$$T_s \equiv \frac{1}{2}\int_S \vec{r} \cdot \rho\vec{v}\vec{v} \cdot d\vec{S}, \tag{2}$$

$$U_s \equiv \frac{1}{2}\int_S P\vec{r} \cdot d\vec{S}, \tag{3}$$

   and

$$\dot{\Psi}_s \equiv \frac{d}{dt}\int_S (\rho\vec{v}r^2) \cdot d\vec{S}. \tag{4}$$

(a) Give a physical explanation for the origin of the $T_s$, $U_s$ and $\dot{\Psi}_s$ terms.

(b) Consider a homogeneous, spherical cloud of mass $M$ and radius $R$. Show that if the time dependent terms in Eq. 1 are zero, and if $T \gg T_s$, then we can construct the following expression for the thermal pressure acting on the surface of the cloud:

$$P_s = \frac{1}{4\pi}\left(-\frac{3}{5}\frac{GM^2}{R^4} + 3\frac{c_s^2 M}{R^3} + \frac{\sigma^2 M}{R^3}\right), \tag{5}$$

where $c_s$ is the isothermal sound speed, and $\sigma$ is the 1D internal velocity dispersion. [Note: $T = (1/2)M\sigma^2$].

(c) We can define a "gravitational pressure" $P_G$ as

$$P_G \equiv -\frac{W}{3V} \tag{6}$$

where $V$ is the volume of the cloud. Show, using Eq. 5, that the total energy of the cloud can be written as

$$E = \frac{3}{2}\left(P_s - P_G\right)V \tag{7}$$

(d) A giant molecular cloud with mass $M = 10^5\ M_\odot$, radius $R = 10$ pc, sound speed $c_s = 0.2\mathrm{kms}^{-1}$ and velocity dispersion $\sigma = 3.0\mathrm{kms}^{-1}$ is surrounded by cold atomic gas with thermal pressure $P/k_\mathrm{B} = 2 \times 10^4$ K cm$^{-3}$, where $k_\mathrm{B}$ is the Boltzmann constant. What is the total energy of the cloud?

3. **Degenerate electron gas** 25 pt

Assume that for a degenerate electron gas all quanum levels up to the Fermi momentum $p_F$ are filled and others are empty, so that the phase-space probability density is given by

$$f(\vec{q},\vec{p}) = \begin{cases} 2/h^3 & for \quad p \le p_F \\ 0 & for \quad p > p_F \end{cases} \tag{8}$$

The factor 2 takes into account the two spin orientations of the electrons.

Use this distribution function to compute the density and pressure and show that this gas has a polytropic equation of state

$$P = K\rho^\gamma \tag{9}$$

Find the index $\gamma$ and the constant $K$, assuming each electron is accompanied by one proton whose kinetic energy is negligible.