Assignment #6: due Tuesday, Dec. 01

Theoretical Astrophysics

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1. **Parker instability** 30 pt

Consider the Galactic disk consists of isothermal gas which is threaded with a horizontal magnetic field. Assume a constant gravitational field perpendicular to the disk plane in the z direction, i.e. $\vec{g} = -\hat{z}\, g$ and a magnetic field parallel to the disk plane x which varies only with z, i.e. $\vec{B} = \hat{x}\, B(z)$. For simplicity study the system in two dimensions using cartesian coordinates.

(a) Assume that the system is in hydrostatic equilibrium with a constant ratio of the magnetic to thermal pressure, i.e.

$$\alpha \equiv \frac{B^2}{8\pi\, P} = \text{const.} \tag{1}$$

What is the pressure distribution as a function of z? Use the relation $P = c_\mathrm{s}^2\, \rho$ where c_s is the constant speed of sound and the scale height $H = (1+\alpha)\, c_\mathrm{s}^2/g$ to express the result.

Now consider this system slightly perturbed out of its equilibrium. Then, from the linear perturbation analysis one gets the following dispersion relation in the xz-plane,

$$n^4 \;+\; c_\mathrm{s}^2 \left[(1+2\alpha)\left(k^2 + \frac{k_0^2}{4}\right)\right] n^2 + k_x^2\, c_\mathrm{s}^4 \left[2\alpha\, k^2 + k_0^2 \left[\left(1 + \frac{3\alpha}{2}\right) - (1+\alpha)^2\right]\right] = 0\,, \tag{2}$$

where $n = i\omega$, $k_0 = H^{-1}$, and the Fourier modes in the x and z direction for the perturbed quantities are

$$\exp\left(i\omega\, t - ik_x\, x\right) \quad , \quad \exp\left(i\omega\, t - ik_z\, z\right) \tag{3}$$

with $k^2 = k_x^2 + k_z^2$. (It is an useful exercise to derive the dispersion relation 2. However, you are not requested to do so in this homework assignment.)

(b) Show that in the absence of a magnetic field all roots (in terms of n^2) of this dispersion relation are negative, i.e. $n^2 < 0$. What is the physical implication of this result regarding the instability?

(c) In the case of a non-vanishing magnetic field derive the instability criterion for the Parker instability, also called magnetic Rayleigh-Taylor instability,

$$\left(\frac{k}{k_0/2}\right)^2 < 2\alpha + 1 \; . \tag{4}$$

Hint: Use the roots of n^2 to find at least one unstable mode, i.e. $n^2 > 0$.

(d) Show that the instability criterion is equivalent to

$$\lambda_x \;>\; \Lambda_x \equiv 4\pi\, H \left[\frac{1}{2\alpha + 1}\right]^{1/2} \tag{5}$$

$$\lambda_z \;>\; \Lambda_z \equiv \frac{\Lambda_x}{\left(1 - (\Lambda_x/\lambda_x)^2\right)^{1/2}} \; , \tag{6}$$

with the wavelengths $\lambda_x = 2\pi/k_x$, and $\lambda_z = 2\pi/k_z$. Discuss this result.

2. **Magnetic Braking of an Alligned Rotor** 30 (+ 10) pt

Consider a uniform gaseous disk of density $\rho_{\rm cl}$ and half-thickness Z rotating rigidly with an initial angular velocity Ω_0. Furthermore assume the disk is threaded with a magnetic field $\vec{B}$ of strength B_0, initially uniform and parallel to the rotation axis. The magnetic field links the disk with the external medium of density $\rho_{\rm ext}$ which is initially at rest. Assume axisymmetry and use cylindrical coordinates (R, φ, z) for the calculations.

(a) Derive the evolution equations for the toroidal magnetic field B_φ and the angular velocity Ω (where $v_\varphi = R\,\Omega$) outside of the disk, i.e. $|z| > Z$, assuming that the radial velocity v_r and the poloidal velocity v_z are negligible small (compared to the Alfvén velocity).

(b) Show that the evolution of the external medium can be expressed by the wave equation

$$\frac{\partial^2 \Omega}{\partial t^2} = v_{\rm A,ext}^2 \, \frac{\partial^2 \Omega}{\partial z^2} \tag{7}$$

where $v_{\rm A,ext} = B_0/\sqrt{4\pi\,\rho_{\rm ext}}$ is the Alfvén velocity in this medium.

(c) Derive the evolution equation of the angular velocity at the surface of the disk, i.e. $|z| = Z$, using the torque per unit area, $N = R\,B_0\,B_\varphi/4\pi$, which the magnetic field exerts on the surface of the disk. Result:

$$\frac{\partial^2 \Omega_{\rm cl}}{\partial t^2} = \frac{1}{Z}\,\frac{\rho_{\rm ext}}{\rho_{\rm cl}}\, v_{\rm A,ext}^2 \, \left.\frac{\partial \Omega}{\partial z}\right|_{|z|=Z} \tag{8}$$

(d) *Bonus:* Combine equations (7) and (8) to calculate the spin-down time of the disk. (+10 pt)

Hint: Use the solution of equation (7) at the disk surface $|z| = Z$.