Assignment #7: due Tuesday, Dec. 8

Theoretical Astrophysics

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1. **A Simple Approach to Magnetorotational Instability** 45 (+10) pt

The simplest system that displays the magnetorotational instability is an axis-symmetric differentially rotating gaseous disk in the presence of a weak vertical magnetic field, i.e. $\vec{B}_0 = B_z \vec{e}_z$. We assume the disk is initially homogeneous with density ρ_0 and has no radial or vertical motions, i.e. $\vec{v}_0 = v_\varphi \vec{e}_\varphi$. In the following we consider a fluid element that is displaced from its appropriate circular orbit by a small distance $\vec{x} = x_R \vec{e}_R + x_\varphi \vec{e}_\varphi$, where the displacement follows a vertical oscillation $\propto \exp(ikz)$. We neglect the effects of viscosity and study the time evolution of the system with perturbations induced as described above. As in the previous assignment we use a cylindrical coordinate system.

(a) Derive the equations of magneto-hydrodynamics for this system to linear order for the quantities $\delta\rho$, δP, $\delta\vec{u}$, and $\delta\vec{B}$, which are perturbed density, pressure, velocity and magnetic field, respectively. Consider the velocity $\vec{u}$ which denotes the deviation of the true fluid velocity $\vec{v}$ at any location from the azimuthal circular velocity $R\Omega(R)\vec{e}_\varphi$,

$$u_R = v_R \ , \qquad u_\varphi = v_\varphi - R\Omega(R) \ , \qquad u_z = v_z \ , \tag{1}$$

and where $\Omega(R)$ is the circular velocity at radius R.
The result is,

$$-\omega\frac{\delta\rho}{\rho} + k\delta u_z = 0 \ , \tag{2}$$

$$-i\omega\delta u_R - 2\Omega\delta u_\varphi - i\frac{kB_z}{4\pi\rho}\delta B_R = 0 \ , \tag{3}$$

$$-i\omega\delta u_\varphi + \frac{\kappa^2}{2\Omega}\delta u_R - i\frac{kB_z}{4\pi\rho}\delta B_\varphi = 0 \ , \tag{4}$$

$$-\omega\delta u_z + k\frac{\delta P}{\rho} = 0 \ , \tag{5}$$

$$-\omega\delta B_R = kB_z\delta u_R \ , \tag{6}$$

$$-i\omega\delta B_\varphi = \delta B_R\frac{d\Omega}{d\ln R} + ikB_z\delta u_\varphi \ , \tag{7}$$

$$\delta B_z = 0 \ , \tag{8}$$

where κ is the epicyclic frequency,

$$\kappa^2 = \frac{1}{R^3}\frac{d(R^4\Omega^2)}{dR} \ , \tag{9}$$

which usually has values in the range $\Omega \leq \kappa \leq 2\Omega$. To close this set of equations we take the usual equation of state,

$$\frac{\delta P}{P} = \gamma \frac{\delta\rho}{\rho} \ . \tag{10}$$

(b) Using equations (2) to (10) relate $\delta\vec{B}$ to the displacement vector $\vec{x}$. (Recall that $d\vec{x}/dt = \vec{u}$ and $d/dt \to i\omega$ in Fourier space.)

(c) We now can substitute the magnetic field terms in the equations of motion. Show that these equations describe the R and φ component of the displacement vector as a set of coupled damped oscillators. Use these equations to find the criterion for instability. Discuss your result.

2. **Stability of magnetized gas clouds** 15 pt

Consider a spherical, self-gravitating gas cloud with homogeneous density which is threaded by a uniform magnetic field of strength $\vec{B}$. Assume that the cloud has a radius of $r = 0.1$ pc and a mean density of $\bar{\rho}_n = 1.67 \times 10^{-20}$ g cm^{-3}.

(a) Calculate the mass-to-flux ratio given by $\mu = M_{\mathrm{cl}}/\Phi$, where M_{cl} is the total mass of the cloud and $\Phi = \pi r^2 |\vec{B}|$ is the total magnetic flux in the scenario under consideration. Assume a magnetic field strength of $B = 30\mu$G.

(b) The critical mass-to-flux ratio for gravitational collapse of the magnetized cloud can be derived as $\mu_{\mathrm{crit}} = 0.13 G^{-1/2}$, where G is the gravitational constant. Calculate the critical magnetic field strength B_{crit} for which the cloud is stable against gravitational collapse.

(c) The ambipolar diffusion timescale is defined as

$$\tau_A = 4\pi\gamma\rho_i\rho_n L^2/B^2 \tag{11}$$

where ρ_i and ρ_n denote the density of the ions and neutrals in the gas, respectively, and L corresponds to the size of the cloud. Assuming that ρ_i is given by $1.15 \times 10^{-26}\mathrm{g\,cm^{-3}} \cdot (\rho_n/[3.85 \times 10^{-19}\mathrm{g\,cm^{-3}}])^{1/2}$ and the friction coefficient is $\gamma = 3.3 \times 10^{13}$ g^{-1} cm^3 s^{-1}, calculate τ_A for the magnetic cloud considered above. What is the corresponding gravitational free-fall timescale $\tau_{\mathrm{ff}} = (3\pi/32\mathrm{G}\rho_n)^{1/2}$ of the system? Discuss the consequences of the difference between the two timescales.