Assignment #3: due Thursday, Nov. 4, 2010

Theoretical Astrophysics

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1. **Equation of hydrostatic balance** (bonus points 10 pt)

   Using the equation of hydrostatic balance, obtain a crude estimate of the central temperature of the Sun. Hint: Approximate the differential operator by the finite difference between the solar surface and center.

2. **Degenerate electron gas** 25 pt

   Consider a degenerate electron gas at zero temperature. That means that all quantum levels up to the Fermi momentum p_F are filled and all others are empty. The phase-space probability density is then given by

   $$f(\vec{q},\vec{p}) = \begin{cases} 2/h^3 & \text{for} \quad p \le p_F \\ 0 & \text{for} \quad p > p_F \end{cases} \tag{1}$$

   The factor 2 takes into account the two spin orientations of the electrons.

   Use this distribution function to compute the density and pressure and show that this gas has a polytropic equation of state

   $$P = K\rho^\gamma \tag{2}$$

   Find the index γ and the constant K for the non-relativistic case. Assume that each electron is accompanied by one proton whose kinetic energy is negligible.

3. **Lane-Emden equation** 30 pt

   In a spherically symmetric system, the equations of hydrostatic equilibrium and Poisson's equation are:

   $$\frac{1}{\rho}\frac{\mathrm{d}P}{\mathrm{d}r} = -\frac{\mathrm{d}\Phi}{\mathrm{d}r} \tag{3}$$

   $$\frac{1}{r^2}\frac{\mathrm{d}}{\mathrm{d}r}\left(r^2\frac{\mathrm{d}\Phi}{\mathrm{d}r}\right) = 4\pi\, G\,\rho \tag{4}$$

   where Φ is the gravitational potential and G Newton's gravitational constant.

(a) Taking $\Phi(r_{\rm surf}) = 0$ and $\rho(r_{\rm surf}) = 0$ at the surface of the star, $r = r_{\rm surf}$, show that for a polytropic equation of state, i.e. $P = K\rho^{(n+1)/n} = K\rho^{\gamma}$, the density in the star ($\Phi < 0$) can be expressed as

$$\rho = \left(\frac{-\Phi}{(n+1)K}\right)^n \tag{5}$$

(b) Substitute this expression into the Poisson equation and show that this then reduces to the *Lane-Emden equation*:

$$\frac{1}{z^2}\frac{\mathrm{d}}{\mathrm{d}z}\left(z^2\frac{\mathrm{d}w}{\mathrm{d}z}\right) + w^n = 0 \tag{6}$$

when written in terms of the variables $w = \Phi/\Phi_c$ and $z = r/r_0$, where Φ_c is the potential at $r = 0$ and r_0 is a characteristic length scale. Find an expression for r_0 in terms of n and K.

(c) Given that there exists a solution of the Lane-Emden equation for the given system, show that the radius R of a non-relativistic degenerate star ($n = 3/2$) is related to its total mass by

$$R \propto M^{-1/3}\,. \tag{7}$$