Assignment #5: due Thursday, Nov. 18, 2010

Theoretical Astrophysics

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1. **Virial Theorem with surface terms** 20 pt

The scalar virial theorem with surface terms included can be written as

$$\frac{1}{2}\ddot{I} = 2(T - T_s) + 2(U - U_s) + W - \frac{1}{2}\dot{\Psi}_s \tag{1}$$

where

$$T_s \equiv \frac{1}{2}\int_S x_j \rho v_j v_i n_i dS = \frac{1}{2}\int_S \vec{r}\cdot\rho\vec{v}\;\vec{v}\cdot\vec{n}dS, \tag{2}$$

$$U_s \equiv \frac{1}{2}\int_S P x_i n_i dS = \frac{1}{2}\int_S P\vec{r}\cdot\vec{n}dS, \tag{3}$$

and

$$\dot{\Psi}_s \equiv \frac{d}{dt}\int_S (\rho x_i r^2) n_i dS = \frac{d}{dt}\int_S (\rho\vec{v}r^2)\cdot\vec{n}dS\ , \tag{4}$$

where $\int_S \dots \vec{n}dS$ is the surface integral with the normal vector $\vec{n}$ pointing outwards and where r is the length of the radius vector to the surface element. All other quantities are defined as in the lecture.

(a) Give a physical explanation for the origin of the T_s, U_s and $\dot{\Psi}_s$ terms.

(b) Consider a homogeneous, spherical cloud of mass M and radius R. Show that if the time dependent terms in (1) are zero, and if $T \gg T_s$, then we can construct the following expression for the thermal pressure acting on the surface of the cloud:

$$P_s = \frac{1}{4\pi}\left(-\frac{3}{5}\frac{GM^2}{R^4} + 3\frac{c_s^2 M}{R^3} + \frac{\sigma^2 M}{R^3}\right), \tag{5}$$

where c_s is the isothermal sound speed, and σ is the 3D internal velocity dispersion. [Note: $T = (1/2)M\sigma^2$].

(c) We can define a "gravitational pressure" P_G as

$$P_G \equiv -\frac{W}{3V} \tag{6}$$

where V is the volume of the cloud. Show, using (5), that the total energy of the cloud can be written as

$$E = \frac{3}{2}\left(P_s - P_G\right)V \tag{7}$$

(d) A giant molecular cloud with mass $M = 10^5\,M_\odot$, radius $R = 10\,\mathrm{pc}$, sound speed $c_s = 0.2$ km s^{-1}, and velocity dispersion $\sigma = 3.0$ km s^{-1} is surrounded by cold atomic gas with thermal pressure $P/k_\mathrm{B} = 2 \times 10^4\,\mathrm{K\,cm^{-3}}$, where k_B is the Boltzmann constant. What is the total energy of the cloud?

2. **Evolution of a supernova remnant** 30 pt

Assume first the supernova remnant in its "adiabatic" phase: all the mass of the remnant is concentrated in a thin shell located at the position of the shock at radius, $r = r_s(t)$, where $M \approx 4\pi r_s^3 \rho_1/3$ with ρ_1 being the density of the interstellar medium. Furthermore, the pressure $P(t)$ interior to the shock can be considered uniform and the equations of motion for the thin shell is then

$$\frac{\mathrm{d}}{\mathrm{d}t}\left(M\dot{r}_s\right) = 4\pi\, r_s^2\, P \tag{8}$$

where $\dot{r}_s \equiv \mathrm{d}r_s/\mathrm{d}t$.

(a) Use the jump conditions for a strong shock of an ideal gas with an adiabatic index $\gamma = 5/3$ to estimate the thickness Δr of the shell in terms of r_s (assume the shell density equals the post-shock density).

(b) Given that in the Sedov phase the total internal energy of the gas in the remnant equals 80% of the explosion energy E_SN, show that the equations of motion have a solution of the form

$$r_s = A\, t^\alpha\,. \tag{9}$$

Find the constants α and A.

Assume now that the shell cools rapidly. Because the cooling rate in the gas is proportional to the density squared there is a phase in the evolution when the thermal energy of the freshly shocked gas can no longer be shared evenly throughout the remnant, as it was in the adiabatic phase discussed above. Most of the energy is radiated away and a cool dense shell forms around the still hot interior. To a good approximation the interior can be described as a hot adiabatic gas bubble of constant mass (again equation of state $P \propto \rho^\gamma$ with $\gamma = 5/3$). The evolution of the blast wave is now driven by the adiabatic expansion of this bubble.

(c) Show that this pressure-driven snow plow phase admits again a solution of the form

$$r_s \propto t^\beta\,, \tag{10}$$

and find the index β.