Assignment #6: due Thursday, Nov. 25, 2010

Theoretical Astrophysics

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1. **Accretion Disk** **total: 60 + 10 pt**

Consider a thin (non-selfgravitating), axisymmetric disk in cylindrical coordinates, (R, θ, z), whose evolution can be described by the continuity equation and the Navier-Stokes equation

$$\frac{\partial \vec{v}}{\partial t} + (\vec{v} \cdot \nabla)\, \vec{v} = \vec{g} - \frac{1}{\rho} \nabla P + \vec{\eta} \tag{1}$$

(here given in Cartesian coordinates), where $\vec{\eta}$ is the viscous friction and $\vec{g}$ is the gravitational acceleration due to the central star with mass M. Assume that the velocity does not vary with disk height z and $v_z = 0$.

(a) Use the surface density $\Sigma = \int \rho \, \mathrm{d}z$ to rewrite the continuity equation in the form

$$\frac{\partial \Sigma}{\partial t} + \frac{1}{R} \frac{\partial}{\partial R} (R \, \Sigma \, v_R) = 0 \tag{2}$$

and derive the equation which determines the evolution of the angular momentum

$$\frac{\partial}{\partial t} \left(\Sigma \, R^2 \, \Omega \right) + \frac{1}{R} \frac{\partial}{\partial r} \left(\Sigma \, R^3 \, \Omega \, v_R \right) = T \tag{3}$$

where $\Omega = v_\theta / R$ is the angular velocity and the function T involves the viscous terms. *(20 + 10 pt)*

(b) Assume that the viscous force per unit area f_{visc} is proportional to the rotational shear $A = R \, \mathrm{d}\Omega / \mathrm{d}R$, i.e.

$$f_{\mathrm{visc}} = \nu \, \rho \, A \tag{4}$$

where ν is the kinetic viscosity, derive the viscous term as

$$T = \frac{1}{R} \frac{\partial}{\partial R} \left(\nu \Sigma R^3 \frac{d\,\Omega}{dR} \right) . \tag{5}$$

(10 pt)

(c) Now assume that the disk matter is in Keplerian orbits around the central star, i.e. $\Omega = (G\,M/R^3)^{1/2}$. Show that the disk evolution is then governed by the equation

$$\frac{\partial\Sigma}{\partial t} = \frac{3}{R}\frac{\partial}{\partial R}\left(R^{1/2}\frac{\partial}{\partial R}(\nu\,\Sigma\,R^{1/2})\right) . \tag{6}$$

(10 pt)

(d) Consider a steady state, thin accretion disk, i.e. $|\nabla P/\rho| \ll G\,M/r^2$, $\partial\vec{v}/\partial t = 0$. Show that the mass infall rate $\dot{M} = -2\pi\,R\,\Sigma\,v_R$ is given by

$$\dot{M} = 3\pi\,\nu\,\Sigma\left[1 - \left(\frac{R_*}{R}\right)^{1/2}\right]^{-1} \tag{7}$$

where R_* are the radii with vanishing shear (say the stellar surface or the inner disk radius). *(10 pt)*

(e) Given that the energy dissipation per unit area of the disk surface is given by

$$\dot{E} = -\nu\,\Sigma\,R^2\left(\frac{\mathrm{d}\Omega}{\mathrm{d}R}\right)^2 , \tag{8}$$

show that the total luminosity emitted from the accretion disk is then

$$L = \frac{G\,M\,\dot{M}}{2\,R_*} . \tag{9}$$

Compare this to the total potential energy released and discuss possible discrepancies. *(10 pt)*