

Assignment #8: due Thursday, Dec. 9, 2010

Theoretical Astrophysics

Winter 2010/2011

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1. Parker instability

30 pt

Consider an isothermal gas in the galactic disk which is threaded with a horizontal magnetic field. Assume a constant gravitational field perpendicular to the disk plane in the z direction, i.e. $\vec{g} = -\hat{z}g$ and a magnetic field parallel to the disk plane x which varies only with z , i.e. $B = \hat{x}B(z)$. For simplicity study the system in two dimensions using cartesian coordinates.

- (a) Assume that the system is in hydrostatic equilibrium with a constant ratio of the magnetic to thermal pressure, i.e.

$$\alpha \equiv \frac{B^2}{8\pi P} = \text{const.} \quad (1)$$

What is the pressure distribution as a function of z ? Use the relation $P = c_s^2 \rho$ where c_s is the constant speed of sound and the scale height $H = (1 + \alpha) c_s^2/g$ to express the result.

Now consider this system slightly perturbed out of its equilibrium. Then, from the linear perturbation analysis one gets the following dispersion relation in the xz -plane,

$$n^4 + c_s^2 \left[(1 + 2\alpha) \left(k^2 + \frac{k_0^2}{4} \right) \right] n^2 + k_x^2 c_s^4 \left[2\alpha k^2 + k_0^2 \left[\left(1 + \frac{3\alpha}{2} \right) - (1 + \alpha)^2 \right] \right] = 0, \quad (2)$$

(it is a good exercise to derive this relation), where $n = i\omega$, $k_0 = H^{-1}$, and the Fourier modes in the x and z direction for the perturbed quantities are

$$\exp(i\omega t - ik_x x) \quad , \quad \exp(i\omega t - ik_z z) \quad (3)$$

with $k^2 = k_x^2 + k_z^2$.

- (b) Show that in the absence of a magnetic field all roots (in terms of n^2) of this dispersion relation are negative, i.e. $n^2 < 0$. What is the physical implication of this result regarding the instability?
- (c) In the case of a non-vanishing magnetic field derive the instability criterion for the Parker instability (magnetic Rayleigh-Taylor instability)

$$\left(\frac{k}{k_0/2} \right)^2 < 2\alpha + 1. \quad (4)$$

Hint: Use the roots of n^2 to find at least one unstable mode, i.e. $n^2 > 0$.

(d) Show that the instability criterion is equivalent to

$$\lambda_x > \Lambda_x \equiv 4\pi H \left[\frac{1}{2\alpha + 1} \right]^{1/2} \quad \text{and} \quad \lambda_z > \Lambda_z \equiv \frac{\Lambda_x}{\left(1 - (\Lambda_x/\lambda_x)^2\right)^{1/2}}, \quad (5)$$

with the wavelengths $\lambda_x = 2\pi/k_x$, and $\lambda_z = 2\pi/k_z$.

2. Plasma Waves: Fluid Treatment

25 pt

A common approximation in plasma physics is that the electrons and/or ions can be described by fluid equations. Assume the ions in a plasma can be treated as a smooth, uniform, motionless background charge density that neutralizes the average electron charge density. The electron number density n and velocity \vec{v} obey the continuity equation and the equation of motion, in which the Lorentz force appears in the same way as an external force,

$$\frac{\partial n}{\partial t} + \vec{\nabla} \cdot (n\vec{v}) = 0, \quad (6)$$

$$\frac{\partial \vec{v}}{\partial t} + (\vec{v} \cdot \vec{\nabla})\vec{v} = \frac{1}{nm_e} \left(-\vec{\nabla}P + qn\vec{E} + \frac{1}{c}\vec{j} \times \vec{B} \right), \quad (7)$$

where m_e is the electron mass, $q = -e$ is the electron charge, P the pressure of the electron gas, \vec{j} the current density, and \vec{E} and \vec{B} are the electric and magnetic field strengths. The electron density and pressure are related by the equation of state,

$$P = Kn^\gamma, \quad (8)$$

where K is a constant and where we assume the electrons behave as an adiabatic gas with $\gamma = 5/3$. Electric and magnetic fields, \vec{E} and \vec{B} , are related to the charge density qn and the current \vec{j} via Maxwell's equations.

(a) Assume the system originally is in equilibrium and apply a small perturbation of the form,

$$\begin{aligned} n &= n_0 + \delta n, & P &= P_0 + \delta P, & \vec{v} &= \delta \vec{v}, \\ \vec{E} &= \delta \vec{E}, & \vec{B} &= \delta \vec{B}, & \vec{j} &= \delta \vec{j}, \end{aligned}$$

where n_0 and P_0 are the homogeneous equilibrium density and pressure. Linearize the equations and find the equation that governs the evolution of the density perturbation δn .

(b) Consider plane wave perturbations of the form $\delta n \propto \exp(i\vec{k}\vec{x} - i\omega t)$, derive their dispersion relation, and find their phase and group velocities as function of the thermal velocity of the electrons, $v_{\text{th}} = (P_0/m_en_0)^{1/2}$, and the ratio k/k_D , where

$$k_D = \left(\frac{4\pi q^2 n_0}{m_e v_{\text{th}}^2} \right)^{1/2}$$

is the Debye wave number as introduced in the lecture.

(c) Discuss the nature of this waves in the limits of small and larger wavelengths.