Assignment #9: due Thursday, Dec. 16, 2010

Theoretical Astrophysics

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In a cold, magnetized plasma consisting of electrons (charge $q_{\rm e} = -e$, mass $m_{\rm e}$) and ions (charge $q_{\rm i} = Z\,e$, mass $m_{\rm i}$), the equation govering the propagation of a wave-like disturbance, $\vec{E} = \vec{E}_0 \exp\left(i\vec{k}\cdot\vec{x} - i\omega t\right)$ is

$$\mathcal{E}\,\vec{E} = 0\,. \tag{1}$$

We use cartesian coordinates with basis $(\vec{e}_x, \vec{e}_y, \vec{e}_z)$ and assume that the wave propagates along the magnetic field which we take parallel to $\vec{e}_z$. In this case, the dielectric tensor $\mathcal{E}$ is

$$\mathcal{E} = \begin{pmatrix} S - n^2 & -iD & 0 \\ iD & S - n^2 & 0 \\ 0 & 0 & P \end{pmatrix}\,, \tag{2}$$

where $n = kc/\omega$ is the refractive index, and

$$S = 1 - \frac{\omega_{\rm pe}^2}{\omega^2 - \Omega_{\rm e}^2} - \frac{\omega_{\rm pi}^2}{\omega^2 - \Omega_{\rm i}^2}\,, \tag{3}$$

$$D = \frac{\omega_{\rm pe}^2\,\Omega_{\rm e}}{\omega(\omega^2 - \Omega_{\rm e}^2)} + \frac{\omega_{\rm pi}^2\,\Omega_{\rm i}}{\omega(\omega^2 - \Omega_{\rm i}^2)}\,, \tag{4}$$

$$P = 1 - \frac{\omega_{\rm pe}^2}{\omega^2} - \frac{\omega_{\rm pi}^2}{\omega^2}\,. \tag{5}$$

The quantities $\omega_{\rm pe,pi} = \sqrt{4\pi\, n_{\rm e,i}\, q_{\rm e,i}^2/m_{\rm e,i}}$ are the electron and ion plasma frequencies (with $n_{\rm e,i}$ the number densities) and $\Omega_{\rm e,i} = q_{\rm e,i}B/m_{\rm e,i}\,c$ are the electron and ion gyration frequencies. Note that both have opposite signs.

1. **Alfvén waves** 30 pt

   (a) Find the dispersion relation in a neutral electron-proton plasma in the low frequency limit, $\omega \ll \Omega_{\rm i}$ and $\omega \ll \omega_{\rm pi}$. Make use of the fact that $m_{\rm i} \gg m_{\rm e}$. Show that only transveral waves are permitted.

   (b) Find the polarization vectors of the corresponding transversal modes. Note, they correspond to the eigenvectors of the system.

2. **Faraday rotation** 30 pt

(a) In the high frequency limit, $\omega \gg \omega_{\mathrm{pe,pi}}$ and $\omega \gg \Omega_{\mathrm{e,i}}$, show that the dispersion relation in the electron-proton plasma for waves travelling in the positive z direction can be written approximately as

$$\frac{k\,c}{\omega} = 1 - \frac{\omega_{\mathrm{pe}}^2 + \omega_{\mathrm{pi}}^2}{2\omega^2} \pm \frac{\omega_{\mathrm{pe}}^2\,\Omega_{\mathrm{e}}}{2\omega^3}\,. \tag{6}$$

The upper and lower signs refer to the polarization vectors $(1/\sqrt{2}, \pm i/\sqrt{2}, 0)$. Use again the fact that $m_{\mathrm{i}} \gg m_{\mathrm{e}}$.

(b) Show that a linearly polarized photon that is emitted along the magnetic field will rotate its direction of polarization as it propagates by an amount proportional to the inverse square of its frequency.

(c) For ionized hydrogen gas in the Galactic plane with $n = 1\,\mathrm{cm}^{-3}$ and $B = 20\,\mu G$, find the distance over which a photon of frequency 3 GHz that is emitted linearly polarized in the x-direction travels before it is converted to one polarized in the y-direction. Assume propagation along a uniform magnetic field.

(d) How does the result change if the photon propagates in a hypothetical electron-positron plasma, where $m_{\mathrm{e}} = m_{\mathrm{i}}$?