Homework Assignment #4 is due Wednesday, Nov. 11, 2015

Theoretical Astrophysics (MKTP2)

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1. **Lane-Emden Equation and Polytropic Stars** *30 pt*

   In a spherically symmetric system, the equations of hydrostatic equilibrium and Poisson's equation are:

$$\frac{1}{\rho}\frac{\mathrm{d}P}{\mathrm{d}r} = -\frac{\mathrm{d}\Phi}{\mathrm{d}r} \tag{1}$$

$$\frac{1}{r^2}\frac{\mathrm{d}}{\mathrm{d}r}\left(r^2\frac{\mathrm{d}\Phi}{\mathrm{d}r}\right) = 4\pi\, G\, \rho \tag{2}$$

   where Φ is the gravitational potential and G Newton's gravitational constant.

   (a) Taking $\Phi(r_{\mathrm{surf}}) = 0$ and $\rho(r_{\mathrm{surf}}) = 0$ at the surface of the star, $r = r_{\mathrm{surf}}$, show that for a polytropic equation of state, i.e. $P = K\rho^{(n+1)/n} = K\rho^{\gamma}$, the density in the star ($\Phi < 0$) can be expressed as

$$\rho = \left(\frac{-\Phi}{(n+1)K}\right)^n \tag{3}$$

   (b) Substitute this expression into the Poisson equation and show that this then reduces to the *Lane-Emden equation*:

$$\frac{1}{z^2}\frac{\mathrm{d}}{\mathrm{d}z}\left(z^2\frac{\mathrm{d}w}{\mathrm{d}z}\right) + w^n = 0 \tag{4}$$

   when written in terms of the variables $w = \Phi/\Phi_c$ and $z = r/r_0$, where Φ_c is the potential at $r = 0$ and r_0 is a characteristic length scale. Find an expression for r_0 in terms of n and K.

   (c) Given that there exists a solution of the Lane-Emden equation for the given system, show that the radius R of a non-relativistic degenerate star ($n = 3/2$) is related to its total mass by

$$R \propto M^{-1/3}\,. \tag{5}$$

2. **Application of the Scalar Virial Theorem** *15 pt*

   (a) Globular clusters are gravitationally bound, dense stellar systems with half-light radii of typically $r_{1/2} \sim 5\,\mathrm{pc}$ and (one-dimensional) velocity dispersions $\sigma_{1/2}$ of roughly $6\,\mathrm{km\,s^{-1}}$. The total bolometric luminosity is $\sim 2.4 \times 10^{38}\,\mathrm{erg\,s^{-1}}$. Using the scalar virial theorem calculate the mass-to-light ratio Υ for a globular cluster in units of the solar mass-to-light ratio $\Upsilon_\odot = M_\odot/L_\odot$.

   (b) The dwarf-spheroidal galaxy Draco - a satellite galaxy of our Milky Way - has a distance of about $d \approx 72\,\mathrm{kpc}$ and an estimated half-mass radius $r_{1/2}$ of about $5.7'$ on the sky. Draco has a luminosity of $L = 2.6 \times 10^5\,L_\odot$ and a (one-dimensional) dispersion velocity of $\sigma \approx 13.2\,\mathrm{km\,s^{-1}}$. Calculate the mass-to-light ratio Υ of the Draco galaxy.

   (c) Compare the result to the one found by Irwin & Hatzidimitriou (1995, Monthly Notices of The Royal Astronomical Society, 277, 1354). Speculate about the difference? The high mass-to-light ratio suggests that Draco is dark matter dominated. Can you think of an alternative explanation for the high value of Υ derived above?